Spinor helicity formalsim and (super)amplitudes of $D = 11$ supergravity and $D = 10$ SYM theory

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Outline

- Recent years we are witnesses of a great progress in amplitude calculations (including multiloop amplitudes; see reviews [Bern, Carrasco, Dixon, Johansson and Roiban, Fortsch.Phys. 2011], [Benincasa, Int.J.Mod.Phys. A 2014], and refs. therein) an important part of which is related to the use of twistor-like and (super)twistor methods, and with BCFW approach first developed for tree gluon amplitudes in [R. Britto, F. Cachazo, B. Feng and E. Witten, PRL2005] (see also [Britto, Cachazo, Feng, NPB05])
- and generalized for tree and loop *super*amplitudes of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG in
  - Arkani-Hamed, Cachazo, Kaplan, JHEP 2010 [arXiv:0808.1446[hep-th]],
  - Brandhuber, Heslop, Travaglini, PRD 2008 [arXiv:0807.4097 [hep-th]].

- The list of important papers in this direction certainly includes
  - Bianchi, Elvang, D. Freedman, JHEP 2008 [arXiv:0805.0757 [hep-th]],
  - Drummond, Henn, Korchemsky, E. Sokatchev, NPB 2010 [arXiv:0807.1095],
  - Drummond, Henn, Plefka, JHEP 2010 [arXiv:0902.2987 [hep-th]],

  and many others... (Sorry for missed references!)

Main elements used in the D=4 superamplitude calculations are, schematically,

- spinor helicity variables (essentially four dimensional!)
- on-shell superfields
- superamplitudes=superfield description of the amplitudes=multiparticle generalization of the on-shell superfields

Higher D generalizations of BCFW

- [Cheung and O'Connell JHEP 2009] generalization to D=6.
- For D=10: [Caron-Huot+ O'Connell JHEP 10]: i) D=10 spinor helicity formalism and ii) "Clifford superfield" description of tree D=10 SYM superamplitudes (quite non minimal and it is not easy to use it).
- The spinor helicity formalism from [Caron-Huot and O'Connell JHEP 2010] was mainly used in the context of type IIB supergravity: [Boels, O'Connell, JHEP 12, Boels PRL 12, Wang, Yin, PRD 15].
- In this talk, based on Phys.Rev.Lett.118(2017) [arXiv:1605.00036], arXiv:1705.nnnnn and [paper in prep.], we describe the generalization of the spinor helicity formalism, on-shell superfield description and the BCFW relations for D=11 SUGRA and D=10 SYM superamplitudes.
- Actually we have proposed (and are elaborating) two approaches
  - Constrained superamplitude formalism
    and
  - almost unconstrained analytic superamplitude formalism.

PRL 2017 [arXiv:1605.00036], [arXiv:1706. in preparation], arXiv:1705.nnnnn

- In more details:
- The staring point of this work was the observation that 10D spinor helicity variables of [Caron-Huot+O'Connell 2010] can be identified with
  - spinor moving frame variables [Bandos, Zheltukhin 91-95], [Bandos, Nurmagambetov 96], ... or, equivalently, with
  - D=10 Lorentz harmonics [Galperin, Howe, Stelle 91, Galperin, Delduc, Sokatchev 91]
    - This observation was made independently in [Uvarov CQG 2016, arXiv:1506.01881] and used their to develop 5D spinor helicity formalism.
- This allowed us
  - to find immediately the spinor helicity formalism for 11D amplitudes
  - to propose a simpler constrained superfield formalism for superamplitudes of D=10 SYM (constrained superfields versus Clifford superfields).
  - and to develop the constrained superamplitude formalism for $D = 11$ SUGRA.
  - To propose the BCFW relations for 10D and 11D superamplitudes
- To find an almost unconstrained analytic superamplitude formalism for $D = 11$ SUGRA and 10D SYM [1705.nnnnn]].

Outline

Bosonic spinors and spinor helicity formalism.

- In the spinor helicity formalism for D=4 amplitudes

$$\mathcal{A}(1,..,n) := \mathcal{A}(p_{(1)}, \varepsilon_{(1)}; ...; p_{(n)}, \varepsilon_{(n)}) = \mathcal{A}(\lambda_{(1)}, \bar{\lambda}_{(1)}; \ldots; \lambda_{(n)}, \bar{\lambda}_{(n)}) \ .$$

the (light-like) momenta $p_{\mu(i)}$ and polarizations of the external particles are described by the bosonic Weyl spinors $\lambda^A_{(i)} = (\bar{\lambda}^{\dot{A}}_{(i)})^*$. In particular,

$$p_{\mu(i)}\sigma^{\mu}_{A\dot{A}} = 2\lambda_{A(i)}\bar{\lambda}_{\dot{A}(i)} \qquad \Leftrightarrow \qquad p_{\mu(i)} = \lambda_{(i)}\sigma_{\mu}\bar{\lambda}_{(i)}, \qquad \mu = 0, ..., 3$$

where $\sigma^{\mu}_{A\dot{A}}$ are relativistic Pauli matrices, $A = 1, 2$, $\dot{A} = 1, 2$, and

$$\sigma^{\mu}{}_{A\dot{A}}\sigma_{\mu B\dot{B}} \equiv 2\epsilon_{AB}\epsilon_{\dot{A}\dot{B}} \quad \Rightarrow \quad p_{\mu i}p^{\mu}_{i} = 0 \ .$$

- Introducing $< ij > \equiv < \lambda_{(i)}\lambda_{(j)} > = \epsilon_{AB}\lambda^A_{(i)}\lambda^B_{(j)}$, $[ij] := [\bar{\lambda}_{(i)}\bar{\lambda}_{(j)}] = \epsilon_{\dot{A}\dot{B}}\bar{\lambda}^{\dot{A}}_{(i)}\bar{\lambda}^{\dot{B}}_{(j)}$
- the simplest MHV amplitude [Parke & Taylor, PRL86] reads

$$\mathcal{A}^{MHV}(1, ..., n) = \delta^4\left(\sum_i \lambda_{A(i)}\bar{\lambda}_{\dot{A}(i)}\right) \frac{< ij >^4}{< 12 > \ldots < (n-1)n >< n1 >}$$

where the i-th and j-th particles are assumed to be of negative helicity.

Helicity

- The amplitude should obey the *helicity constraints*,

$$\hat{h}_{(i)}\mathcal{A}(1,...,n) = h_i\mathcal{A}(1,...,n) \; ,$$

where h_i is the helicity of the state, $h_i = \pm 1$ in the case of gluons, and

$$2\hat{h}_{(i)} := -\lambda^A_{(i)}\frac{\partial}{\partial\lambda^A_{(i)}} + \bar{\lambda}^{\dot{A}}_{(i)}\frac{\partial}{\partial\bar{\lambda}^{\dot{A}}_{(i)}} \; .$$

- Thus the n-particle amplitudes are also characterized by n helicities. For gluons these are ± 1 and the amplitude carries n sign indices,

$$\mathcal{A}(1,...,n) = \mathcal{A}^{-\cdots-\cdots+\cdots+}(1,...,n).$$

- It can be shown that $\mathcal{A}^{+\cdots+}(1,...,n) = 0$, $\mathcal{A}^{-+\cdots+}(1,...,n) = 0$,
- so that the simplest *maximal helicity violation (MHV)* amplitude is

$$\mathcal{A}^{MHV}(1,...,n) = \mathcal{A}^{+\cdots+-_i+\cdots+-_j+\cdots+}(1,...,n) = \frac{\delta^4\left(\sum_i \lambda_{A(i)}\bar{\lambda}_{\dot{A}(i)}\right) < ij >^4}{< 12 > \ldots < n1 >}$$

BCFW deformations

- The BCFW recursion relations

$$\mathcal{A}_n = \sum_{\mathcal{I},h} \mathcal{A}_{\mathcal{I}}^{h} \frac{1}{P_{\mathcal{I}}{}^2} \mathcal{A}_{\mathcal{J}}^{-h}, \qquad \textit{where} \qquad \mathcal{I} \bigcup \mathcal{J} = (1, ..., n)$$

use the on-shell amplitudes depending on the deformed spinors, say

$$\lambda^A_{(n)} \mapsto \widehat{\lambda^A_{(n)}} = \lambda^A_{(n)} + z\lambda^A_{(1)}, \qquad \bar{\lambda}^{\dot{A}}_{(n)} \mapsto \widehat{\bar{\lambda}^{\dot{A}}_{(n)}} = \bar{\lambda}^{\dot{A}}_{(n)},$$

$$\lambda^A_{(1)} \mapsto \widehat{\lambda^A_{(1)}} = \lambda^A_{(1)} , \qquad \bar{\lambda}^{\dot{A}}_{(1)} \mapsto \widehat{\bar{\lambda}^{\dot{A}}_{(1)}} = \bar{\lambda}^{\dot{A}}_{(1)} - z\bar{\lambda}^{\dot{A}}_{(n)},$$

- which implies the deformation of 1st and n-th momenta

$$p^a_{(n)} \mapsto \widehat{p^a_{(n)}}(z) = p^a_{(n)} + zq^a , \qquad p^a_{(1)} \mapsto \widehat{p^a_{(1)}}(z) = p^a_{(1)} - z\bar{q}^a ,$$

$$q^a q_a = 0 , \qquad p^a_{(n)} q_a = 0 , \qquad p^a_{(1)} q_a = 0 .$$

The deformed momenta are generically complex but remain light-like,

$$\widehat{p^a_{(n)}}\, \widehat{p_{(n)a}} = 0 , \qquad \widehat{p^a_{(1)}}\, \widehat{p_{(1)a}} = 0 .$$

BCFW recurrent relations. Explicit form.

- The BCFW recurrent relations for tree amplitudes of D=4 gluons read

$$\mathcal{A}^{(n)}(p_1, p_2, \ldots ; p_n) = \sum_h \sum_l^n \mathcal{A}_h^{(l+1)}(\widehat{p}_1(z_l); p_2; \ldots ; p_l; \widehat{P_{\Sigma_l}}(z_l)) \times \frac{1}{(P_{\Sigma_l})^2} \mathcal{A}_{-h}^{(n-l+1)}(-\widehat{P_{\Sigma_l}}(z_l), p_{l+1}; \ldots ; \widehat{p}_n(z_l)) \; ,$$

where h is the helicity of intermediate state with $\widehat{P_{\Sigma_l}}(z_l)$,

$$P^a_{\Sigma_l} = -\sum_{m=1}^{l} p^a_m \qquad and \qquad \widehat{P^a_{\Sigma_l}}(z) = -\sum_{m=1}^{l} \widehat{p^a_m}(z)$$

- $\sum_l$ is the sum over l and over distributions of particles among $\mathcal{A}_{\pm h}^{\{(l+1) \atop (n-l+1)}$.
- The specific l-dependent value of the complex parameter z,

$$z_l := P^a_{\Sigma_l} P_{\Sigma_l a} / 2 P^b_{\Sigma_l} q_b$$

- is such that $\boxed{(\widehat{P^a_{\Sigma_l}}(z_l))^2 = 0}$ $\Rightarrow$ r.h.s. contains on-shell amplitudes.

Superamplitudes and on-shell superfields for $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA

- One can also collect the n-particle amplitudes of the fields of SYM (SUGRA) in the superfield amplitude (superamplitude)

$$\mathcal{A}(1;...;n) = \mathcal{A}(\lambda_{(1)}, \bar{\lambda}_{(1)}, \eta_{(1)}; ...; \lambda_{(n)}, \bar{\lambda}_{(n)}, \eta_{(n)}) \; ,$$

depending on n fermionic $\eta^q_{(i)} = (\bar{\eta}_{q(i)})^*$ in fundamental rep. of $SU(4)$ ($SU(8)$), $q = 1, ..., 4$ (...8).

- This is possible because the on-shell states of the maximal SYM (SUGRA) multiplet can be collected in an **on-shell superfield**

$$\Phi(\lambda, \bar{\lambda}, \eta^q) = f^{(-s)} + \eta^q \chi_q + \frac{1}{2}\eta^q\eta^p s_{pq} + \ldots + \frac{1}{\mathcal{N}!}\eta^q_1 \ldots \eta^{q_{\mathcal{N}}} \epsilon_{q_1 \ldots q_{\mathcal{N}}} f^{(+s)} \; ,$$

chiral superfield on an *on-shell superspace* of super-helicity $s = \frac{\mathcal{N}}{4}$,

$$\boxed{\hat{h}\Phi(\lambda, \bar{\lambda}, \eta^q) = s\Phi(\lambda, \bar{\lambda}, \eta^q)} \; , \qquad \hat{h} := -\frac{1}{2}\lambda^A \frac{\partial}{\partial \lambda^A} + \frac{1}{2}\bar{\lambda}^{\dot{A}} \frac{\partial}{\partial \bar{\lambda}^{\dot{A}}} + \frac{1}{2}\eta^q \frac{\partial}{\partial \eta^q} \; .$$

- The $\mathcal{N}=4$ (8) superamplitudes obey n superhelicity constraints

$$\hat{h}_{(i)}\mathcal{A}(\{\lambda_{(j)}, \bar{\lambda}_{(j)}, \eta^q_{(j)}\}) = s\mathcal{A}(\{\lambda_{(j)}, \bar{\lambda}_{(j)}, \eta^q_{(j)}\}) \; , \qquad s = \frac{\mathcal{N}}{4} \; .$$

The power of superamplitudes for $\mathcal{N}=4$ SYM

- SUSY Ward identities $Q_{Aq}\mathcal{A}(\{\lambda_{(i)}, \bar\lambda_{(i)}, \eta^q_{(i)}\}) = 0, \;\; \bar{Q}^q_A\mathcal{A}(...) = 0$ immediately imply that

$$\mathcal{A}^{+\cdots+}(\{\lambda_{(i)}, \bar\lambda_{(i)}\}) = \int \prod_i d^{\mathcal{N}}\eta_i \; \mathcal{A}(\{\lambda_{(i)}, \bar\lambda_{(i)}, \eta^q_{(i)}\}) = 0 \qquad \textit{and}$$

$$\mathcal{A}^{+\cdots+-}(\{\lambda_{(i)}, \bar\lambda_{(i)}\}) = \int \prod_{i=1}^{n-1} d^{\mathcal{N}}\eta_i \; d^{\mathcal{N}}\bar\eta_n \; \mathcal{A}(\eta_1, ..., \eta_{(n-1)}, \bar\eta_n) = 0$$

- and also fix the form of tree MHV superamplitude of $\mathcal{N}=4$ SYM

$$\mathcal{A}^{MHV}(\{\lambda_{(i)}, \bar\lambda_{(i)}, \eta^q_{(i)}\}) =\propto \frac{\delta^4(\sum_i \lambda_i \sigma^a \bar\lambda_{(i)})\delta^{2\mathcal{N}}(\sum_i \lambda^A_{(i)}\eta^q_{(i)})}{< 12 > ... < n1 >}$$

[Nair, PLB 1988].

- The MHV amplitude can be obtained from MHV superamplitude as

$$\mathcal{A}^{+\cdots+--}(\{\lambda_{(i)}, \bar\lambda_{(i)}\}) = \int \prod_{i=1}^{n-2} d^{\mathcal{N}}\eta_i \; d^{\mathcal{N}}\bar\eta_{(n-1)} \; d^{\mathcal{N}}\bar\eta_n \; \mathcal{A}(\eta_1, ..., \bar\eta_{(n-1)}, \bar\eta_n)$$

BCFW relations for superamplitudes

- In the BCFW-like recurrent relations for tree superamplitudes of $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity [Brandhuber, Heslop, Travaglini, PRD 2008, Arkani-Hamed, Cachazo, Kaplan, JHEP 2010].

$$\mathcal{A}^{(n)}(k_1,\eta_1;\ldots;k_n,\eta_n) = \\ = \sum_l \int d^{\mathcal{N}}\eta \mathcal{A}^{(l+1)}_{z_l}(\widehat{k}_1,\widehat{\eta}_1;k_2,\eta_2;\ldots;k_l,\eta_l;\widehat{P_{\Sigma_l}}(z_l),\eta)\frac{1}{(P_{\Sigma_l})^2}\times \\ \times\mathcal{A}^{(n-l+1)}_{z_l}(-\widehat{P_{\Sigma_l}}(z_l),\eta;k_{l+1},\eta_{(l+1)};\ldots;k_{n-1},\eta_{n-1};\widehat{k}_n,\widehat{\eta}_n)\ .$$

- the deformations of the bosonic spinors

$$\widehat{\lambda^A_{(n)}} = \lambda^A_{(n)} + z\lambda^A_{(1)}, \qquad \widehat{\bar{\lambda}^{\dot{A}}_{(1)}} = \bar{\lambda}^{\dot{A}}_{(1)} - z\bar{\lambda}^{\dot{A}}_{(n)},$$

- is supplenented by the deformation of fermionic $\eta^q = (\bar{\eta}_q)^*$,

$$\widehat{\eta^q_{(n)}}(z) = \eta^q_{(n)} + z\eta^q_{(1)}\ , \qquad \widehat{\eta^q_{(1)}}(z) = \eta^q_{(1)}\ .$$

- New issues (w/r to bosonic BCFW): i) $\sum_h \mapsto \int d^{\mathcal{N}}\eta$, and ii) $\widehat{\eta^q_{(n)}}(z) = \eta^q_{(n)} + z\eta^q_{(1)}$ which mixes gluon and gluino amplitudes.

Outline

Spinor moving frame in D=11

- In D=4: $p_{\mu(i)}\sigma^{\mu}_{A\dot{A}} = 2\lambda_{A(i)}\bar{\lambda}_{\dot{A}(i)} \quad \Leftrightarrow \quad p_{\mu(i)} = \lambda_{(i)}\sigma_{\mu}\bar{\lambda}_{(i)}$.
- Similarly, the light-like k_a of a massless 11D particle can be expressed by

$$\boxed{k_a\Gamma^a_{\alpha\beta} = 2\rho^{\#} v^{-}_{\alpha q} v^{-}_{\beta q}}\, , \qquad \boxed{\rho^{\#} v^{-}_{q} \tilde{\Gamma}_a v^{-}_{p} = k_a \delta_{qp}}\, ,$$

in terms of 'energy variable' $\rho^{\#}$ and

- a set of 16 **constrained** bosonic 32-component spinors $\boxed{v^{-}_{\alpha q}}$, $q, p = 1, ..., 16$, $\alpha = 1, ..., 32$ which can be identified with
  - D=11 spinor moving frame variables [Bandos, Zheeltukhin 92, Bandos 2006-2007]
  - 11D Lorentz harmonics [Galperin, Howe, Townsend NPB 93].
- Essentially, the constraints on $v^{-}_{\alpha q}$ are given by the above equations supplemented by $\boxed{v^{-}_{\alpha q} C^{\alpha\beta} v^{-}_{\beta p} = 0}$,
- and by the requirement that the rank of 32×16 matrix $v^{-}_{\alpha q}$ is = 16.

Spinor moving frame variables in D=11

- One can show that (roughly speaking) in the theory with local $SO(1,1) \otimes SO(9)$ symmetry, $\boxed{v_{\alpha q}^{\;-}}$ obeying the above constraints

$$u_a^{=}\Gamma^a_{\alpha\beta} = 2\rho^{\#} v_{\alpha q}^{\;-} v_{\beta q}^{\;-} \;, \qquad v_q^{-}\tilde{\Gamma}_a v_p^{-} = u_a^{=}\delta_{qp} \;, \qquad v_{\alpha q}^{\;-} C^{\alpha\beta} v_{\beta q}^{\;-} = 0$$

$(u_a^{=} \equiv k_a/\rho^{\#})$ can be considered as homogeneous coordinates on $\mathbb{S}^9$, the celestial sphere of a D=11 observer,

$$\boxed{\{v_{\alpha q}^{\;-}\} = \mathbb{S}^9} \;. \qquad \left(\mathbb{S}^9 = \frac{SO(1,10)}{[SO(1,1) \otimes SO(9)] \subset\!\!\!\!\times K_9} \right)$$

Spinor moving frame and spinor helicity formalism

- One can check that, due to the above constraints the momentum k_a $(= \rho^{\#} u_a^{=})$ is light-like $\qquad \boxed{k_a k^a = 0}$
- and that $v_{\alpha q}^{\;-}$ and $v_q^{-\alpha} = -iC^{\alpha\beta} v_{\beta q}^{\;-}$ obey the Dirac equations

$$k_a \tilde{\Gamma}^{a\,\alpha\beta} v_{\beta q}^{\;-} = 0 \qquad \Leftrightarrow \qquad k_a \Gamma^a_{\alpha\beta} v_q^{-\beta} = 0 \;.$$

11D Spinor helicity formalism

- The 11D counterpart of the 10D spinor helicity variables of Caron-Huot and O'Connell are $\lambda_{\alpha q} = \sqrt{\rho^{\#}} v_{\alpha q}^{-}$;
- the 11D counterpart of the polarization spinor of the fermionic field is $\lambda_q^{\alpha} = \sqrt{\rho^{\#}} v_q^{-\alpha} = -iC^{\alpha\beta}\lambda_{\beta q}$ $(= (\lambda_q^{\alpha})^*)$.
- The constraints on $v_{\alpha q}^{-}$ can be written in terms of λ_{α}

$$k_a \Gamma^a_{\alpha\beta} = 2\lambda_{\alpha q}\lambda_{\beta q}, \qquad \lambda_q \tilde{\Gamma}_a \lambda_p = k_a \delta_{qp} \qquad \lambda C \lambda = 0$$

- Then why we need $\rho^{\#}$ and $v_{\alpha q}^{-} = \lambda_{\alpha q}/\sqrt{\rho^{\#}}$?
  - The geometric and group theoretic meaning of $v_{\alpha q}^{-}$ is much more clear.
  - $\rho^{\#}$ and its canonically conjugate coordinate $x^{=}$ will play an important role in the construction of on-shell superfields and superamplitudes.
- In particular the D=11 counterpart of the on-shell superspace is

$$\Sigma^{(10|16)}: \qquad \{(x^{=}, v_{\alpha q}^{-}; \theta_q^{-})\} ,$$

with bosonic sector $\mathbb{R} \otimes \mathbb{S}^9$ including $\mathbb{R} = \{x^{=}\}$ and $\mathbb{S}^9 = \{v_{\alpha q}^{-}\}$.

Brink-Schwarz superparticle and spinor moving frame

- But where such seemingly strange spinor frame variables come from?
- To understand this it is useful to discuss massless superparticle model.
- To start from Brink-Schwarz action which does exist in any dimensions and to follow the way to the Ferber-Schirafuji-like spinor moving frame formulation.
- The quantization of superparticle in its spinor moving frame formulation leads us to an appropriate on-shell superfield formalism which can be then generalized to superamplitudes.
- Here we just briefly describe the results of this procedure
- starting from a few more details on spinor frame

Vector frame attached to light-like momentum

- A particular solution of the mass shell conditions $k_a k^a = 0$ is given by

$$k_a = \rho\,(1, 0, ..., 0, -1)$$

- Any other solution can be obtained from this (or from the reflected one) by performing a τ-dependent $O(1, D-1)$ Lorentz transformation

$$u_a^{(b)}(\tau) \; \in \; O(1, D-1) \quad \Leftrightarrow \quad u_a^{(b)} u^{a(c)} = \eta^{(b)(c)} = diag(+1, -1, ..., -1)\,,$$

- so that the general solution of the mass-shell constraint $p_a p^a = 0$ is

$$k_a = u_a^{(b)} p_{(b)} = \rho(u_a^0 - u_a^{(D-1)}) =: \rho^{\#}(\tau) u_a^{=}(\tau)\,,$$

- By construction, the vector $u_a^{=}(\tau) = (u_a^0 - u_a^{(D-1)})$ is light-like.
- It is convenient to write the frame matrix $u_a^{(b)}(\tau) \in SO(1, D-1)$ in terms of this $u_a^{=}$, its complementary light-like $u_a^{\#}(\tau) = (u_a^0 + u_a^{(D-1)})$ and u_a^I,

$$u_a^{(b)} = \left(\frac{1}{2}\left(u_a^{=} + u_a^{\#}\right),\; u_a^I\,,\, \frac{1}{2}\left(u_a^{\#} - u_a^{=}\right) \right) \; \in \; SO^{\uparrow}(1, D-1)\,.$$

Frame variables= Vector Lorentz harmonics

- The defining relation for the **moving frame matrix** or the matrix of **vector Lorentz harmonics** (or light-cone harmonics) [Sokatchev 86]

$$u_a^{(b)} = \left(\frac{1}{2}\left(u_a^{=} + u_a^{\#}\right),\ u_a^{I},\ \frac{1}{2}\left(u_a^{\#} - u_a^{=}\right) \right) \ \in \ SO^{\uparrow}(1, D-1)\,,$$

is equivalent to $u_a^{(b)} u^{a(c)} = \eta^{(a)(c)}$ (see [E. Sokatchev, 86,87]), i.e.

$$\begin{aligned} & u_a^{=} u^{a=} = 0\,, \\ & u_a^{\#} u^{a\#} = 0\,, \quad u_a^{=} u^{a\#} = 2\,, \\ & u_a^{I} u^{a=} = 0 = u_a^{I} u^{a\#}\,, \quad u_a^{I} u^{aJ} = -\delta^{IJ} \end{aligned}$$

$$\textit{and} \qquad \delta_a{}^{b} = \frac{1}{2} u_a^{=} u^{b\#} + \frac{1}{2} u_a^{\#} u^{b=} - u_a^{I} u^{bI}\,.$$

- Resuming: a frame can be attached to a light-like momentum by setting

$$\boxed{k_a = \rho^{\#} u_a^{=}}\ .$$

Moving frame variables=$SO(1, D-1)/[SO(1,1) \otimes SO(D-2)] \otimes K_{D-2}$

- The splitting of $u_a^{(b)}$ is manifestly invariant under $SO(1,1) \times SO(D-2)$ so that in a model with this gauge symmetry the vector harmonics= homogeneous coordinates of the coset $\frac{SO(1,D-1)}{SO(1,1)\times SO(D-2)}$

$$\{u_a^{=}, u_a^{\#}, u_a^{I}\} \; = \; \frac{SO(1,D-1)}{SO(1,1)\times SO(D-2)}$$

(E.g. moving frame formulation of superstring [Bandos, Zheltukhin 91,92])

- In the model involving only $u_a^{=}$ (massless superparticle), the gauge symmetry increases to $[SO(1,1) \times SO(D-2)] \otimes K_{D-2}$ where K_{D-2} is

$$u_a^{=} \mapsto u_a^{=}, \qquad u_a^{\#} \mapsto u_a^{\#} + \tfrac{1}{4} u_a^{=} (K^{\# I})^2 + u_a^{I} K^{\# I},$$
$$u_a^{I} \mapsto u_{a(i)}^{I} + \tfrac{1}{2} u_{a(i)}^{=} K^{\# I} \,,$$

- and the set of harmonic variables parametrize a compact coset

$$\boxed{\{(u_a^{=}, u_a^{\#}, u_a^{I})\} \; = \; \frac{SO(1,D-1)}{[SO(1,1)\times SO(D-2)]\otimes K_{D-2}} = \mathbb{S}^{D-2}} \quad or \quad \{u_a^{=}\} = \mathbb{S}^{D-2}$$

[Galperin, Howe, Stelle 91, Galperin, Delduc, Sokatchev 91].

Spinor moving frame = $\sqrt{\textit{moving frame}}$

- Spinor moving frame = $\sqrt{\textit{moving frame}}$ is defined by conditions of Lorentz invariance of D-dimensional Γ^a and also $C_{\alpha\beta}$ if such exists,
- i.e. is defined by a matrix $V \in Spin(1, D-1)$ which obeys

$$V\Gamma_b V^T = u_b^{(a)}\Gamma_{(a)} \, , \qquad V^T\tilde{\Gamma}^{(a)}V = \tilde{\Gamma}^b u_b^{(a)} \; ,$$
$$VCV^T = C \, , \qquad for \; D \; in \; \; which \quad \exists C \, .$$

- The $SO(1,1) \times SO(D-2)$ invariant splitting of the spinor moving frame matrix, corresponding to $u_b^{(a)} = (u_b^{=}, u_b^{\#}, u_b^{I})$, is

$$V_\alpha^{(\beta)} = \begin{pmatrix} v_{\alpha\dot{q}}^{+}, & v_{\alpha q}^{-} \end{pmatrix} \in Spin(1, D-1) \, ,$$

where q and $\dot{q}$ are indices of the spinor representations of $SO(D-2)$, which can be different, like s-spinor and c-spinor in D=10,

$$D = 10 : \qquad \alpha = 1, ..., 16 \, , \quad \dot{q} = 1, ..., 8 \, , \quad q = 1, ..., 8 \, ,$$

or the same, as in D=11,

$$D = 11 : \qquad \alpha = 1, ..., 32 \, , \quad q = \dot{q} = 1, ..., 16 \, , \quad v_{\alpha\dot{q}}^{+} \equiv v_{\alpha q}^{+} \, .$$

Inverse spinor moving frame matrix

- The rectangular blocks of the spinor moving frame matrix, $v_{\alpha q}^{\;-}$ and $v_{\alpha\dot{q}}^{\;+}$ are called **spinor moving frame variables** or **spinor harmonics** (spinorial Lorentz harmonics).
- When the charge conjugation matrix exists, the elements of the inverse spinor moving frame matrix

$$V_{(\beta)}{}^{\alpha} = \begin{pmatrix} v_q^{+\alpha} \\ v_{\dot{q}}^{-\alpha} \end{pmatrix} \in Spin(1, D-1)$$

can be costructed from the harmonics $v_{\alpha\dot{q}}^{\;+}$ and $v_{\alpha q}^{\;-}$. For instance,

$$D = 11 \;\; : \qquad v_{\alpha\, q}^{\;-} = iC_{\alpha\beta} v_q^{-\beta} \; , \qquad v_{\alpha\, q}^{\;+} = -iC_{\alpha\beta} v_q^{+\beta} \; .$$

- When the charge conjugation matrix does not exist, like it is in $D = 10$ (MW representation), these are defined by the conditions

$$v_q^{+\alpha} v_{\alpha p}^{\;-} = \delta_{qp} \; , \qquad v_q^{+\alpha} v_{\alpha\dot{p}}^{\;+} = 0 \; ,$$
$$v_{\dot{q}}^{-\alpha} v_{\alpha q}^{\;-} = 0 \; , \qquad v_{\dot{q}}^{-\alpha} v_{\alpha\dot{p}}^{\;+} = \delta_{\dot{q}\dot{p}} \; ,$$

or equivalently, $V_{\alpha}^{(\beta)} V_{(\beta)}{}^{\gamma} := v_{\alpha}^{-\dot{q}} v_{\dot{q}}^{+\gamma} + v_q^{-\alpha} v_q^{-\gamma} = \delta_{\alpha}^{\gamma}$.

- With the suitable representation for Γ–matrices, the constraints $V\Gamma_b V^T = u_b^{(a)}\Gamma_{(a)}$ $V^T\tilde{\Gamma}^{(a)}V = \tilde{\Gamma}^b u_b^{(a)}$ can be split into

$$u_a^{=}\Gamma^a_{\alpha\beta} = 2v_{\alpha q}{}^{-}v_{\beta q}{}^{-}\ , \qquad v_q^{-}\tilde{\Gamma}_a v_p^{-} = u_a^{=}\delta_{qp},$$
$$,\qquad u_a^{\#}\Gamma^a_{\alpha\beta} = 2v_{\alpha\dot{q}}{}^{+}v_{\beta\dot{q}}{}^{+}\ , \qquad v_{\dot{q}}^{+}\tilde{\Gamma}_a v_{\dot{p}}^{+} = u_a^{\#}\delta_{\dot{q}\dot{p}}\ ,$$
$$u_a^{I}\Gamma^a_{\alpha\beta} = 2v_{(\alpha|q}{}^{-}\gamma^I_{q\dot{q}}v_{|\beta)\dot{q}}{}^{+}\ , \quad v_q^{-}\tilde{\Gamma}_a v_{\dot{p}}^{+} = u_a^I\gamma^I_{q\dot{p}}\ .$$

- For D=11 $q,p\equiv\dot{q},\dot{p} = 1,...,16$ are spinor indices of SO(9) and $\gamma^I_{qp} = \gamma^I_{pq}$ is the SO(9) gamma matrix;
- for D=10 $\gamma^I_{p\dot{q}} =: \tilde{\gamma}^I_{\dot{q}p}$ are Klebsh-Gordan coefficients of SO(8), $q,p = 1,...,8$ are s-spinor (8s) indices, $\dot{q},\dot{p} = 1,...,8$ are c-spinor (8c) indices and I=1,.., 8 is SO(8) vector index (8v-index).
- In our perspective the especially important among above relations are

$$u_a^{=}\Gamma^a_{\alpha\beta} = 2v_{\alpha q}{}^{-}v_{\beta q}{}^{-}\ , \qquad v_q^{-}\tilde{\Gamma}_a v_p^{-} = u_a^{=}\delta_{qp}$$

- which allow to state that $v_{\alpha q}^{-}$ is a square root of $u_a^{=}$
- in the same sense as in D=4 one states λ_A "=" $\sqrt{p_a}$ $(p_\mu\sigma^\mu_{A\dot{A}} = 2\lambda_A\bar{\lambda}_{\dot{A}})$.

D=10 vs D=11 spinor frame formalism

- In D=10 the above should be completed by eqs for inverse harmonics,

$$u_a^{=}\tilde{\Gamma}^{a\,\alpha\beta} = 2v_{\dot{q}}^{-\alpha}v_{\dot{q}}^{-\beta}\;, \qquad v_{\dot{q}}^{-}\Gamma_a v_{\dot{p}}^{-} = u_a^{=}\delta_{\dot{q}\dot{p}}\;,$$
$$\tilde{\Gamma}^{a\alpha\beta}u_a^{\#} = 2v_q^{+\alpha}v_q^{+\beta}\;, \qquad v_q^{+}\Gamma_a v_p^{+} = u_a^{\#}\delta_{qp}\;,$$
$$v_{\dot{q}}^{-}\Gamma_a v_p^{+} = -u_a^{I}\gamma^{I}_{p\dot{q}}\;, \qquad 2v_{\dot{q}}^{-(\alpha}\gamma^{I}_{q\dot{q}}v_q^{+\beta)} = -\tilde{\Gamma}^{a\alpha\beta}u_a^{I}\;.$$

while for D=11 $v_{\dot{q}}^{-\alpha} \equiv v_q^{-\alpha} = -iC^{\alpha\beta}v_{\beta q}^{-}$ and these equations are not independent.

D=10 vs D=11 spinor helicity formalism

- The D=10 spinor helicity variables of Caron-Huot and O'Connell is $\lambda_{\alpha q} = \sqrt{\rho^{\#}}v_{\alpha q}^{-}$ carrying 8s index, while the polarization spinor is $\lambda_{\dot{q}}^{\alpha} = \sqrt{\rho^{\#}}v_{\dot{q}}^{-\alpha}$ which carries 8c spinor index of SO(8).
- This is in contrast to 11D, where the polarization vector actually coincides with the spinor helicity variable $\lambda_q^{\alpha} = \sqrt{\rho^{\#}}v_q^{-\alpha} = -iC^{\alpha\beta}\lambda_{\beta q}$.

On shell fields of D=10 SYM in spinor frame form of spinor helicity formalism

- The polarization vector of the vector field can be identified with u_a^I so that the on-shell field strength of the (D=10) gauge field

$$D=10: \quad F_{ab}=k_{[a}u_{b]}{}^I\, w^I \;, \qquad a=0,1,...,9\;, \qquad I=1,...,8$$

  is characterized by an SO(8) vector w^I.
- The linearized on-shell spinor field

$$D=10: \quad \chi^\alpha = v_{\dot q}^{-\alpha}\psi_{\dot q}\;, \qquad \dot q = 1,...,8\;,$$

  is characterized by a fermionic SO(8) c-spinor $\psi_{\dot q}$.
- The on-shell d.o.f.'s of SYM $\leftrightarrow$ $w^I=w^I(\rho^\#, v_{\alpha q}^-)$, $\psi_{\dot q}=\psi_{\dot q}(\rho^\#, v_{\alpha q}^-)$ or, making Fourier transform w/r to $\rho^\#$, $w^I(x^=, v_q^-)$ and $\psi_q(x^=, v_q^-)$.
- Supersymmetry acts on these 9d fields by

$$\delta_\epsilon \psi_{\dot q} = \epsilon^{-q}\gamma^I_{q\dot q}\, w^I \;, \qquad \delta_\epsilon w^I = 2i\epsilon^{-q}\gamma^I_{q\dot q}\partial_= \psi_{\dot q},$$

$$\textit{where} \qquad \epsilon^{-q}=\epsilon^\alpha v_{\alpha q}^-\;.$$

On shell fields of D=11 SUGRA in spinor frame/spinor helicity formalism

- The linearized on-shell field strength of 3-form gauge field

$$D=11: \quad F_{abcd} = k_{[a}u_b{}^I u_c{}^J u_{d]}{}^K \, \Phi_{IJK} \,, \qquad a=0,1,...,10, \quad I=1,...,9\,,$$

is expressed in terms of antisymmetric SO(9) tensor Φ_{IJK} $(= A_{IJK})$.

- Its superpartners γ–traceless Ψ_{Iq} and traceless h_{IJ}, are used to make a decomposition of linearized on-shell 11D graviton and gravitino fields,

$$D=11: \qquad \psi^{\alpha}_{ab} = k_{[a}u^I_{b]} v_q^{-\alpha}\Psi_{Iq} \,, \qquad \gamma^I_{qp}\Psi_{Ip} = 0\,,$$
$$h_{ab} = u^I_{(a}u^J_{b)} h_{IJ} \,, \qquad h_{II} = 0\,.$$

These fields will appear as independent components of a constrained on-shell superfield.

Outline

On-shell superfield description of D=10 SYM

- The main on-shell superfield of D=10 SYM is [A. Galperin, P. Howe, P. Townsend NPB1993] a fermionic c-spinor superfield $\Psi_{\dot{q}}$ obeying

$$D^+_q \Psi_{\dot{q}} = \gamma^I_{q\dot{q}} V^I \, , \qquad q = 1, ..., 8 \, , \quad \dot{q} = 1, .., 8 \, , \quad I = 1, .., 8 \, .$$

- The consistency of this eq. requires

$$D^+_q V^I = 2i\gamma^I_{q\dot{q}} \partial_{=} \Psi_{\dot{q}} \, .$$

- $\Rightarrow$ there are no other independent components in the constrained on-shell superfield $\Psi_{\dot{q}}(x^{=}, \theta^-_q, v_{\alpha q}{}^-)$, but $\psi_{\dot{q}} = \psi_{\dot{q}}|_0$ and $w^I = V^I|_0$.

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Indeed,

$$\Psi_{\dot{q}}(x^=, v^-_q; \theta^-_q) = \psi_{\dot{q}}(x^=, v^-_q) + \theta^-_q \gamma^I_{q\dot{q}} w^I(x^=) +$$

$$+ \sum_{k=1}^{4} \left(-\frac{i}{4}\right)^k \frac{(2k-1)!!}{(2k)!! \, (2k)!} (\theta^- \gamma^{I_{k-1} I_k} \theta^-) \ldots (\theta^- \gamma^{I_1 I_2} \theta^-) \, (\gamma^{I_1 I_2} \ldots \gamma^{I_{k-1} I_k})_{\dot{q}\dot{p}} (\partial_=)^k \psi_{\dot{p}} +$$

$$+ \sum_{k=1}^{3} \left(-\frac{i}{4}\right)^k \frac{(2k)!!}{(2k+1)!! \, (2k+1)!} (\theta^- \tilde{\gamma}^{I_1 I_2} \theta^-) \ldots (\theta^- \tilde{\gamma}^{I_{k-1} I_k} \theta^-) (\tilde{\gamma}^{I_1 I_2} \ldots \tilde{\gamma}^{I_{k-1} I_k} \tilde{\gamma}^I \theta^-)_{\dot{q}} (\partial_=)^k w^I.$$

On-shell superfields of 11D SUGRA

- In [A. Galperin, P. Howe, P. Townsend NPB1993] the linearized 11D supergravity was described by a bosonic superfield $\Phi^{IJK} = \Phi^{[IJK]}(x^{=}, \theta^{-}_{q}, v_{\alpha q}{}^{-})$ which obeys

$$D^{+}_{q}\Phi^{IJK} = 3i\gamma^{[IJ}_{qp}\Psi^{K]}_{p}\,, \qquad \gamma^{I}_{qp}\Psi^{I}_{p} = 0\,, \qquad \begin{cases} I,J,K = 1,...,9 \\ q,p = 1,...,16 \end{cases}$$

where $\gamma^{I}_{qp} = \gamma^{I}_{pq}$ are d=9 Dirac matrices, $\gamma^{I}\gamma^{J} + \gamma^{J}\gamma^{I} = \delta^{IJ}\mathbb{I}_{16\times 16}$, and

$$D^{+}_{q} = \partial^{+}_{q} + 2i\theta^{-}_{q}\partial_{=} \equiv \frac{\partial}{\partial\theta^{-}_{q}} + 2i\theta^{-}_{q}\frac{\partial}{\partial x^{=}}$$

obeying the d=1, $\mathcal{N} = 16$ supersymmetry algebra

$$\{D^{+}_{q}, D^{+}_{p}\} = 4i\delta_{qp}\partial_{=}$$

On-shell superfield equations of linearized D=11 SUGRA

- The consistency of $D^+_q \Phi^{IJK} = 3i\gamma^{[IJ}_{qp}\Psi^{K]}_p$ requires, besides $\gamma^I_{qp}\Psi^I_p = 0$, that

$$D^+_q \Psi^I_p = \frac{1}{18}\left(\gamma^{IJKL}_{qp} + 6\delta^{I[J}\gamma^{KL]}_{qp}\right)\partial_= \Phi^{JKL} + 2\partial_= H_{IJ}\gamma^J_{qp} \ ,$$

  with symmetric traceless $SO(9)$ tensor superfield $H_{IJ} = H_{((IJ))}$, obeying

$$D^+_q H_{IJ} = i\gamma^{(I}_{qp}\Psi^{J)}_p , \qquad H_{IJ} = H_{JI}, \quad H_{II} = 0 \ .$$

- These superfield equations (actually any of these three) can be considered as a counterpart of helicity constraint $\hat{h}\Phi = h\Phi$ imposed on the D=4 on-shell superfield.

On-shell superfield equations of linearized D=11 SUGRA

- Back to D=11 supergravity, we find convenient to collect all the on-shell superfields in one object

$$\Psi_Q(x^{=}, v_{\alpha q}^{-}; \theta_q^{-}) = \{\Psi_{Iq}\,, \Phi_{[IJK]}\,,\, H_{((IJ))}\,\}\,,$$

with multiindex Q taking 128(=144-16) 'fermionic' and 128=84+44 'bosonic values',

$$Q = \{Iq\,, [IJK]\,, ((IJ))\,\}$$

(gamma-tracelessness and tracelessness are implied!),

- and to write all the equations for them,

$$D_q^+\Psi_p^I = \frac{1}{3}\left(\gamma_{qp}^{IJKL} + 6\delta^{I[J}\gamma_{qp}^{KL]}\right)\partial_{=}\Phi^{JKL} + 2\partial_{=}H_{IJ}\gamma_{qp}^{J}\,,$$

$$D_q^+\Phi^{IJK} = 3i\gamma_{qp}^{[IJ}\Psi_p^{K]}\,, \qquad D_q^+H_{IJ} = i\gamma_{qp}^{(I}\Psi_p^{J)},$$

in the unique form

$$D_q^+\Psi_Q = \Delta_{Q\,qP}\Psi_P.$$

Fourier transform of the linearized 11D SUGRA equations

- After making Fourier transform

$$\Psi_Q(\rho^{\#}, v^-_{\alpha q}; \theta^-_q) = \frac{1}{2\pi}\int dx^{=}\ \exp(i\rho^{\#}x^{=})\,\Psi_Q(x^{=}, v^-_{\alpha q}; \theta^-_q)$$

- the superfields obey the same $D^+_q\Psi_Q = \Delta_{Q\,qP}\Psi_P$ but with $\partial_{=} \mapsto -i\rho^{\#}$,

$$D^+_q = \partial^+_q + 2\rho^{\#}\theta^-_q \ .$$

- As all $\Delta_{Q\,qP}$ are now algebraic, passing to Fourier image makes natural to choose the fermionic superfield and its equation as fundamental

$$\boxed{D^+_q\Psi^I_p = -\frac{i\rho^{\#}}{3}\left(\gamma^{IJKL} + 6\delta^{I[J}\gamma^{KL]}\right)_{qp}\Phi^{JKL} - 2i\rho^{\#}H_{IJ}\gamma^J_{qp}}\ .$$

- We can define our 11D superamplitudes by generalization of this equation,
- but more convenient way is to start from one of the bosonic superamplitudes.

Outline

10D superamplitudes

- The on-shell n-particle superamplitudes are functions on a direct product of n copies of the on-shell superspace.
- The basic superamplitude of 10D SYM

$$\mathcal{A}^{(n)}_{I_1\ldots I_n}(k_1,\theta_1^-;\ldots;k_n,\theta_n^-)\equiv\mathcal{A}^{(n)}_{I_1\ldots I_n}(\rho_1^{\#};v_{q1}^-;\theta_{q1}^-;\ldots;\rho_n^{\#};v_{qn}^-;\theta_{qn}^-)\ ,$$

carry n bosonic, **8v** indices of SO(8) and obeys

$$\boxed{D^+_{qj}\mathcal{A}^{(n)}_{I_1\ldots I_j\ldots I_n}=2\rho_j^{\#}\,\gamma^{I_j}{}_{q\dot{q}}\mathcal{A}^{(n)}_{I_1\ldots I_{j-1}\dot{q}I_{j+1}\ldots I_n}}\ ,\qquad D^+_{qj}=\frac{\partial}{\partial\theta^-_{qj}}+2\rho_j^{\#}\theta^-_{qj}\ .$$

- Selfconsistency of this equation requires equations for $\mathcal{A}^{(n)}_{I_1\ldots I_{j-1}\dot{q}I_{j+1}\ldots I_n}$ and for amplitudes with higher number of fermions.
- It is convenient to introduce a notation with multi-indices $Q_j=\{\dot{q}_j,I_j\}$ and resume all these equations in one

$$D^+_{qj}\mathcal{A}_{Q_1\ldots Q_j\ldots Q_j}=(-)^{\Sigma_j}\Delta_{Q_j\,qP_j}\mathcal{A}_{Q_1\ldots P_j\ldots Q_j}.$$

- $\Delta_{Q_j\,qP_j}$ can be read off the equations for on-shell superfields, $\Delta_{Iq\dot{q}}=2\rho_j^{\#}\,\gamma^{I_j}{}_{q\dot{q}}$ etc.

11D superamplitudes

- The on-shell n-particle scattering amplitudes of 11D SUGRA

$$\mathcal{A}^{(n)}_{Q_1\ldots Q_n}(k_1,\theta_1^-;\ldots;k_n,\theta_n^-)\equiv \mathcal{A}^{(n)}_{Q_1\ldots Q_n}(\rho_1^{\#};v_{q1}^-;\theta_{q1}^-;\ldots;\rho_n^{\#};v_{qn}^-;\theta_{qn}^-)\;, \qquad ,$$

carry n multi-indices $Q_l=\{I_lq_l\,,[I_lJ_lK_l]\,,((I_lJ_l))\,\}$ and obey

$$\gamma^{I_l}_{p_lq_l}\mathcal{A}_{\ldots I_{(l)}q_{(l)}\ldots}=0,$$
$$D^+_{q(l)}\mathcal{A}_{\ldots Q_{(l)}\ldots}=(-)^{\Sigma_l}\Delta_{Q_l\,qP_{(l)}}\mathcal{A}_{\ldots P_{(l)}\ldots},$$

- $\Delta_{Q_j\,qP_j}$ can be read off eqs. for on-shell superfields,
- and $\Sigma_l=\#$ of fermionic, I_jq_j, indices among $Q_1,\ldots Q_{(l-1)}$, *i.e.*

$$\Sigma_l=\sum_{j=1}^{l-1}\frac{(1-(-)^{\varepsilon(Q_j)})}{2}\;,\qquad \begin{cases}\varepsilon([I_jJ_jK_j])=0=\varepsilon(\,((I_jJ_j))\,)\;,\\ \varepsilon(I_jq_j)=1\;.\end{cases}$$

- In particular, when $Q_l=I_lp_l$, this equation reads

$$(-)^{\Sigma_l}D^{+(l)}_{q_l}\mathcal{A}^{(n)}_{Q_1\ldots\,I_lp_l\,\ldots Q_n}=-i\rho^{\#}_{(l)}\gamma_{J_l\,qp}\mathcal{A}^{(n)}_{Q_1\ldots((I_lJ_l))\ldots Q_n}-$$
$$-\tfrac{i}{18}\rho^{\#}_{(l)}\left(\gamma^{I_lJ_lK_lL_l}_{qp}+6\delta^{I_l[J_l}\gamma^{K_lL_l]}_{qp}\right)\mathcal{A}^{(n)}_{Q_1\ldots[J_lK_lL_l]\ldots Q_n}\;.$$

Generalized BCFW deformations in D=11

- As in 4D construction the deformation implies the shifts

$$\widehat{k^a_{(1)}} = k^a_{(1)} - zq^a \,, \qquad \widehat{k^a_{(n)}} = k^a_{(n)} + zq^a \,, \qquad z \in \mathbb{C},$$
$$q_a q^a = 0 \,, \qquad q_a k^a_{(1)} = 0 \,, \qquad q_a k^a_{(n)} = 0 \,,$$

- In D=11 and D=10 that results from

$$\widehat{v^-_{\alpha q(n)}} = v^-_{\alpha q(n)} + z\, v^-_{\alpha p(1)}\, \mathbb{M}_{pq} \sqrt{\rho^{\#}_{(1)}/\rho^{\#}_{(n)}} \,,$$
$$\widehat{v^-_{\alpha q(1)}} = v^-_{\alpha q(1)} - z\, \mathbb{M}_{qp}\, v^-_{\alpha p(n)} \sqrt{\rho^{\#}_{(n)}/\rho^{\#}_{(1)}}$$

where $\mathbb{M}_{qp} = -2\, q^a \,(v^-_{q(1)}\, \tilde{\Gamma}_a v^-_{p(n)}) \sqrt{\rho^{\#}_{(1)}\rho^{\#}_{(n)}/(k_{(1)}k_{(n)})}$ is nilpotent

$$\boxed{\mathbb{M}_{rp}\mathbb{M}_{rq} = 0}\,, \boxed{\mathbb{M}_{qr}\mathbb{M}_{pr} = 0}\,.$$

- This nilpotent matrix enters also the deformation of the fermionic

$$\widehat{\theta^-_{p(n)}} = \theta^-_{p(n)} + z\,\theta^-_{q(1)}\, \mathbb{M}_{qp} \sqrt{\rho^{\#}_{(1)}/\rho^{\#}_{(n)}} \,,$$
$$\widehat{\theta^-_{q(1)}} = \theta^-_{q(1)} - z\, \mathbb{M}_{qp}\, \theta^-_{p(n)} \sqrt{\rho^{\#}_{(n)}/\rho^{\#}_{(1)}} \,.$$

11D BCFW

BCFW-type recurrent relations for tree 11D superamplitudes [PRL 2017] are

$$\mathcal{A}^{(n)}_{Q_1\ldots Q_n}(k_1,\theta^-_{(1)};k_2,\theta^-_{(2)};\ldots;k_n,\theta^-_{(n)}) =$$
$$= \sum_{l=2}^{n} \frac{(-)^{\Sigma_{(l+1)}}}{64(\widehat{\rho}^{\#}(z_l))^2} D^+_{q(z_l)} \Big(\mathcal{A}^{(l+1)}_{z_l\, Q_1\ldots Q_l\, Jp}(\widehat{k}_1,\widehat{\theta^-_{(1)}};k_2,\theta^-_{(2)};\ldots;k_l,\theta^-_{(l)};\widehat{P}_l(z_l),\theta^-) \times$$
$$\times \frac{1}{(P_l)^2} \overleftrightarrow{D}^+_{q(z_l)} \mathcal{A}^{(n-l+1)}_{z_l\, Jp\, Q_{l+1}\ldots Q_n}(-\widehat{P}_l(z_l),\theta^-;k_{l+1},\theta^-_{(l+1)};\ldots;k_{n-1},\theta^-_{(n-1)};\widehat{k}_n,\widehat{\theta^-_{(n)}}) \Big)\Big|_{\theta^-=0} .$$

- where $P^a_l = -\sum_{m=1}^{l} k^a_m$, $\widehat{P}^a_l(z) = -\sum_{m=1}^{l<n} \widehat{k}^a_m(z) = P^a_l - zq^a$ and
  $\boxed{z_l := \frac{P^a_l P_{l\,a}}{2P^b_l q_b}}$ with q^a obeying $q^2=0$, $q\cdot k_1 = 0$, $q\cdot k_n = 0$
- Actually, $q^a = -\sqrt{\rho^{\#}_1 \rho^{\#}_n} v^-_{q(1)} \tilde{\Gamma}^a \mathbb{M}_{qp} v^-_{p(n)}/32$ with $\mathbb{M}\mathbb{M}^T = 0$.
- Actually, the bosonic arguments of the on-shell amplitudes are $\rho^{\#}_{(i)}$ and $v^-_{\alpha q(i)}$ from $k_{a(i)}\Gamma^a_{\alpha\beta} = 2\rho^{\#}_{(i)} v^-_{\alpha q(i)} v^-_{\beta q(i)}$ and $v^-_{q(i)}\tilde{\Gamma}^a v^-_{p(i)} = k_{a(i)}\delta_{qp}/\rho^{\#}_{(i)}$.

11D BCFW

$$\mathcal{A}^{(n)}_{Q_1\ldots Q_n}(k_1,\theta^-_{(1)};k_2,\theta^-_{(2)};\ldots;k_n,\theta^-_{(n)}) =$$

$$= \sum_{l=2}^{n} \frac{(-)^{\Sigma_{(l+1)}}}{64(\widehat{\rho}^{\#}(z_l))^2} D^+_{q(z_l)} \Big(\mathcal{A}^{(l+1)}_{z_l\; Q_1\ldots Q_l\; Jp}(\widehat{k}_1,\widehat{\theta^-_{(1)}};k_2,\theta^-_{(2)};\ldots;k_l,\theta^-_{(l)};\widehat{P}_l(z_l),\theta^-) \times$$

$$\times \frac{1}{(P_l)^2} \overleftrightarrow{D}^+_{q(z_l)} \mathcal{A}^{(n-l+1)}_{z_l\; Jp\; Q_{l+1}\ldots Q_n}(-\widehat{P}_l(z_l),\theta^-;k_{l+1},\theta^-_{(l+1)};\ldots;k_{n-1},\theta^-_{(n-1)};\widehat{k}_n,\widehat{\theta^-_{(n)}}) \Big)\Big|_{\theta^-=0} .$$

- Actually, the bosonic arguments of the on-shell amplitudes are $\rho^{\#}_{(i)}$ and $v^-_{\alpha q(i)}$ from $k_{a(i)}\Gamma^a_{\alpha\beta} = 2\rho^{\#}_{(i)} v^-_{\alpha q(i)} v^-_{\beta q(i)}$ and $v^-_{q(i)}\tilde{\Gamma}^a v^-_{p(i)} = k_{a(i)}\delta_{qp}/\rho^{\#}_{(i)}$.
- and $\pm\widehat{P}_l{}^a(z_l)$ should be also understood as $v^-_{\alpha q P_l}(z_l)$ and $\pm\rho^{\#}_{P_l}(z_l)$

$$\widehat{P_l{}^a}(z_l)\Gamma_{a\alpha\beta} = 2\rho^{\#}_{P_l}\, v^-_{\alpha q P_l} v^-_{\beta q P_l}\ , \qquad \widehat{P}_l{}^a(z_l)\delta_{qp} = \rho^{\#}_{P_l}\, v^-_{q\,P_l}\tilde{\Gamma}^a v^-_{p\,P_l}\ .$$

- Finally, $D^+_{q(z_l)}$ is the covariant derivative with respect to θ^-_q, $D^+_{q(z_l)} = \frac{\partial}{\partial\theta^-_q} + 2\rho^{\#}_{P_l}\theta^-_q$.

Outline

Little group SO(D-2)$\mapsto$ SO(D-4) tiny group

- Actually there exists a possibility to construct an alternative, analytic superfield formalism [hep-th/1705.nnnn].
- The price to pay is breaking (spontaneous) of the little group symmetry $SO(D-2)_i$ to the 'tiny group' $SO(D-4)$ ($\in SU(\mathcal{N})$).
- The analytic superamplitudes have a superfield structure very similar to its D=4 cousin, but with 'component' amplitudes depending on another set of bosonic variables. These are:
- D=10 or D=11 spinor helicity variables: densities $\rho_i^{\#}$ and $v_{\alpha qi}^{-}$

$$\{v_{\alpha qi}^{-}\} = \left(\frac{Spin(1,D-1)}{[SO(1,1)\otimes Spin(D-2)]\otimes\!\!\!\!\!\!\times K_{D-2}}\right)_i ,$$

and internal frame or internal harmonic variables

$$\{w_{qi}^{A}, \bar{w}_{Aqi}\} = \left(\frac{Spin(D-2)}{Spin(D-4)\otimes U(1)}\right)_i$$

[Harmonic variables, SU(2)/U(1), SU(3)/(U(1)XU(1)),... :
[Galperin, Ivanov, Kalitsin, Ogievetsky, Sokatchev=GIKOS CQG 84,84],
[Galperin, Ivanov, Sokatchev, "Harmonic superspace", CUP 2001],

$\frac{SO(D-2)}{SO(D-4)\otimes U(1)}$ harmonic variables

- This internal frame or internal harmonic variables

$$\{w^A_{qi}, \bar{w}_{Aqi}\} = \left(\frac{Spin(D-2)}{Spin(D-4)\otimes U(1)}\right)_i ,$$

  obey

$$\bar{w}_{qB} w_q{}^A = \delta_B{}^A , \qquad w_q{}^A w_q{}^B = 0 , \qquad \bar{w}_{qA}\bar{w}_{qB} = 0 .$$

  besides

$$\psi_{q\dot{p}} := \gamma^I_{q\dot{p}} U_I = 2\bar{w}_{qA} w^A_{\dot{p}} , \qquad \bar{\psi}^{\check{J}}_{q\dot{p}} := \gamma^I_{q\dot{p}} \bar{U}_I = 2 w^A_q \bar{w}_{\dot{p}A} .$$

  and $\psi^{\check{J}}_{q\dot{p}} := \gamma^I_{q\dot{p}} U^{\check{J}}_I = i w^A_q \sigma^{\check{J}}_{AB} w^B_{\dot{p}} + i\bar{w}_{qA}\tilde{\sigma}^{\check{J}AB}\bar{w}_{\dot{p}B}$ (in D=11 $\dot{q}=q$)
- (in D=11 $\dot{q}=q$, $Spin(7)\subset SU(8)$; for D=10 $Spin(D-4)=SU(4)$).
- Here U_I, $\bar{U}_I$ and $U_I{}^{\check{J}}$ form the vector internal frame

$$U^{(J)}_I = \left(U_I{}^{\check{J}}, \tfrac{1}{2}\left(U_I + \bar{U}_I\right), \tfrac{1}{2i}\left(U_I - \bar{U}_I\right)\right) \in SO(D-2) .$$

Analytic superamplitude of 10D SYM

- We start with the basic $\mathcal{A}^{(n)}_{I_1\ldots I_j\ldots I_n}$ obeying

$$D^{+(j)}_{q_j}\mathcal{A}^{(n)}_{I_1\ldots I_j\ldots I_n} = 2\rho^{\#}_j\,\gamma^{I_j}_{q_j\dot{q}_j}\mathcal{A}^{(n)}_{I_1\ldots I_{j-1}\dot{q}_j I_{j+1}\ldots I_n} \quad :$$

- First, we contract $SO(8)_i$ 8v indices with $U_{I\,i}$ ($\gamma^I_{q\dot{p}}U_{I\,i} = 2\bar{w}_{qAi}w^A_{\dot{p}i}$)

$$\tilde{\mathcal{A}}_n(\{\rho^{\#}_{(i)}, v^-_{\alpha q(i)}; w_i, \bar{w}_i; \theta^-_{qi}\}) = U_{I_1 1}\ldots U_{I_n n}\,\mathcal{A}_{I_1\ldots I_n}(\{\rho^{\#}_i, v^-_{\alpha qi}; \theta^-_{qi}\}) \ ,$$

- we obtain the object which obeys

$$\bar{D}^{+(j)}_A\tilde{\mathcal{A}}_n(\{\rho^{\#}_{(i)}, v^-_{\alpha q(i)}; w_i, \bar{w}_i; \theta^-_{qi}\}) = 0 \qquad \forall j = 1, ..., n \ ,$$

$$\bar{D}^{+(j)}_A = \bar{w}_{qAj}D^{+(j)}_q = \frac{\partial}{\partial\bar{\eta}^{-A}_j} + 2\rho^{\#}_j\eta^-_{Aj} \ , \qquad \eta^-_{Aj} = \theta^-_{qj}\bar{w}_{qAj} = (\bar{\eta}^{-A}_j)^* \ .$$

- Our analytic 10D SYM superamplitude is related to this by

$$\mathcal{A}_n(\{\rho^{\#}_i, v^-_{\alpha qi}; w_i, \bar{w}_i; \eta_{Ai}\}) = e^{-2\sum_j \rho^{\#}_j\eta^-_{Bj}\bar{\eta}^{-B}_j}\,\tilde{\mathcal{A}}_n(\{..., \bar{w}_i;\ \eta^-_{Ai}w^A_{qi} + \bar{\eta}^{-A}_i\bar{w}_{qAi}\})$$

Analytic superamplitude of 11D SUGRA

- The analytic superamplitudes of 11D SUGRA are constructed as

$$\mathcal{A}_n(\{\rho^{\#}_i, v^{-}_{\alpha q i}; w_i, \bar{w}_i; \eta_{Ai}\}) = U_{I_1 1} U_{J_1 1} \ldots U_{I_n n} U_{J_n n} \times$$
$$\times e^{-2\sum_j \rho^{\#}_j \eta^{-}_{Bj} \bar{\eta}^{-B}_j} \mathcal{A}^{(n)}_{(I_1 J_1)\ldots(I_j J_j)\ldots(I_n J_n)}(\{\rho^{\#}_i, v^{-}_{\alpha q i};\ \eta^{-}_{Ai} w^{A}_{qi} + \bar{\eta}^{-A}_i \bar{w}_{qAi}\}).$$

from the basic 11D superamplitude $\mathcal{A}^{(n)}_{I_1 \ldots I_j \ldots I_n}$ obeying

$$D^{+}_{qj} \mathcal{A}^{(n)}_{(I_1 J_1)\ldots(I_j J_j)\ldots(I_n J_n)} = \rho^{\#}_j \gamma_{qp(I_j|} \mathcal{A}^{(n)}_{(I_1 J_1)\ldots(I_{j-1} J_{j-1})\,|J_j)p\,(I_{j+1} J_{j+1})\ldots(I_n J_n)}$$

- Notice that, despite the similarity of the superfield structure of analytic superamplitudes with ones of D=4 $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA
- the generalization of 4D results to 10D and 11D is not straightforward.
- This issue is under investigation now.
- In particular, we have found a Lorentz covariant counterpart of the light cone gauge, fixed on spinor frame variables, which promises to be very useful tool for development of both constrained and analytic superamplitude formalisms.

Outline

Discussion and conclusion

- I hope this study have convinced you that the D=10, 11 Lorentz harmonic approach and(or) spinor moving frame formalism [Galperin, Howe, Stelle 91, Galperin, Delduc, Sokatchev 91, Bandos, Zheltukhin 91-95, Galperin, Howe, Stelle 93, Bandos, Nurmagambetov 96, Bandos, Sorokin, ..., Uvarov,...], which, in contrast to Newmen-Penrose diad and Penrose twistor formalism, work(s) with highly constrained set of spinors,
- is useful, besides the in the superembedding approach
    - [Bandos, Pasti, Sorokin, Tonin, Volkov 95, Bandos, Sorokin, Volkov 95, Howe, Sezgin 96, Howe, Sezgin, West 97, Bandos, Sorokin, Tonin 97, ...]

  also in the on-shell amplitude calculations.
- Of course, we are at the first stages of developing such an application.
- Namely we have constructed/presented:
    - the 10D and 11D spinor helicity formalism,
    - on-shell superfield description of 11D SUGRA and 10D SYM amplitudes = constrained superamplitude formalism
    - the BCFW-type relation for 11D superamplitudes (and a hopefully more convenient version of BCFW for 10D SYM superamplitudes),
- and almost unconstrained analytic superamplitudes formalism.

Outlook

The natural directions for further development are:

- Generalization of constrained and analytic superamplitude approaches to loop (super)amplitudes.
- To develop the spinor moving frame and on-shell superfield approaches to the CHY scattering equations
  [Cachazo, He, Yuan, PRL 2014= arXiv:1307.2199]
  and 'ambitwistor string'
  [Mason, Skinner JHEP 2013, ..., Geyer, Lipstein, Mason PRL14, ..., Adamo, ..., Lipstein, Schomerus, ...]
  (our approach implies rather Green –Schwarz type ambitwistor superstring≈twistor superstring [Ig Bandos, JHEP 14, arXiv:1404.1299]).
- Possible generalization to 10D superstring amplitudes
- (including field theory amplitudes beyond 10D SYM/SUGRA).
- ? 11D superamplitudes beyond 11D SUGRA? (?M-theory amplitudes?)

THE END!

THANK YOU FOR YOUR ATTENTION!

Outline

The 10D SYM superamplitude with four fermionic outcomes can be reproduced from

$$\mathcal{A}_{\dot{q}_1\dot{q}_2\dot{q}_3\dot{q}_4}(k_1,\theta^-_{(1)};k_2,\theta^-_{(2)};k_3,\theta^-_{(3)};k_4,\theta^-_{(4)}) = \\ = \frac{1}{16(\widehat{\rho}^{\#}(z_{12}))^2}\left(D^+_{q(z_{12})}\left(\mathcal{A}_{z_{12}\ \dot{q}_1\dot{q}_2\dot{p}}(\widehat{k_1},\widehat{\theta^-_{(1)}};k_2,\theta^-_{(2)};\widehat{P_{12}}(z_{12}),\Theta^-)\ \times \right.\right.\\ \left.\left.\times\frac{1}{(P_{12})^2}\overleftrightarrow{D}^+_{q(z_{12})}\mathcal{A}_{z_{12}\ \dot{p}\dot{q}_3\dot{q}_4}(-\widehat{P_{12}}(z_{12}),\Theta^-;k_3,\theta^-_{(3)};\widehat{k_4},\widehat{\theta^-_{(4)}})\right)\right)_{\Theta^-_q=0} - (2\leftrightarrow 3$$

After applying the covariant derivatives, this expression can be written in the form

$$\mathcal{A}_{\dot{q}_1\dot{q}_2\dot{q}_3\dot{q}_4}(k_1,\theta^-_{(1)};k_2,\theta^-_{(2)};k_3,\theta^-_{(3)};k_4,\theta^-_{(4)}) = \\ = \quad 2\,\mathcal{A}_{z_{12}\ \dot{q}_1\dot{q}_2\dot{p}}(\widehat{k_1},\widehat{\theta^-_{(1)}};k_2,\theta^-_{(2)};\widehat{\rho^{\#}}(z_{12}),v^-_q(z_{12}),0) \\ \times\frac{1}{(P_{12})^2\widehat{\rho}^{\#}(z_{12})}\mathcal{A}_{z_{12}\ \dot{p}\,\dot{q}_3\dot{q}_4}(-\widehat{\rho^{\#}}(z_{12}),v^-_q(z_{12}),0;k_3,\theta^-_{(3)};\widehat{k_4},\widehat{\theta^-_{(4)}}) \\ +\mathcal{A}_{z_{12}\ \dot{q}_1\dot{q}_2\ l}(\widehat{k_1},\widehat{\theta^-_{(1)}};k_2,\theta^-_{(2)};\widehat{\rho^{\#}}(z_{12}),v^-_q(z_{12}),0) \\ \times\frac{1}{(P_{12})^2}\mathcal{A}_{z_{12}\ l\,\dot{q}_3\dot{q}_4}(-\widehat{\rho^{\#}}(z_{12}),v^-_q(z_{12}),0;k_3,\theta^-_{(3)};\widehat{k_4},\widehat{\theta^-_{(4)}}) + (2\longleftrightarrow 3)\ .$$

From this we can obtain the BCFW relation for the 4-fermionic amplitude of 10D SYM

$$\mathcal{A}_{\dot{q}_1\dot{q}_2\dot{q}_3\dot{q}_4}(k_1;k_2;k_3;k_4) =$$
$$= \quad \mathcal{A}_{z_{12}\ \dot{q}_1\dot{q}_2\ l}(\widehat{k_1};k_2,;\widehat{\rho^{\#}}(z_{12}),v_q^-(z_{12})) \times \frac{1}{(P_{12})^2}\mathcal{A}_{z_{12}\ l\,\dot{q}_3\dot{q}_4}(-\widehat{\rho^{\#}}(z_{12}),v_q^-(z_{12});k_3,;\widehat{k_4}) + (2 \longleftrightarrow 3)\ .$$

Its structure is simpler than that of superamplitude
because the amplitudes of odd number of fermions vanishes, in particular

$$\mathcal{A}_{z_{12}\ \dot{q}_1\dot{q}_2\ \dot{p}}(\widehat{k_1};k_2,;\widehat{\rho^{\#}}(z_{12}),v_q^-(z_{12})) \equiv 0$$

(which is not the case for superamplitudes).

Outline

It is convenient to introduce an auxiliary spinor frame $(v^-_{\alpha q}, v^+_{\alpha q})$ and associated vector frame $(u^{=}_a, u^{\#}_a, u^I_a)$. Then

- any of the spinor and vector frames $(v^-_{\alpha q(i)}, v^+_{\alpha q(i)})$ and associated vector frame $(u^{=}_{a(i)}, u^{\#}_{a(i)}, u^I_{a(i)})$ associated to one of the scattered particles are related to these by the Spin(1,D-1) Lorentz transformations
- but only $(D-2)$ of the parameters of this Lorentz transformation, $K_i^{=I}$ $(\approx \mathbb{S}^{D-2)})$, are not associated to gauge symmetry which defines spinor frame(s)
- thus we can fix the gauge $(K_i^{\#I}=0\ ,\ \mathcal{O}_i^{IJ}=\delta^{IJ},\ e^{-\beta_i}=1)$ in which any spinor frame can be expressed through the auxiliary frame by

$$v^-_{\alpha q(i)} = v^-_{\alpha q} + \frac{1}{2} K_i^{=I} \gamma^I_{qp} v^+_{\alpha p}\ , \qquad v^+_{\alpha q(i)} = v^+_{\alpha q}\ .$$

- The frame vectors are related to the vectors of auxiliary frame by

$$u^{=}_{a(i)} = u^{=}_a + K^{=I}_{(i)} u^I_a + \frac{1}{4} (\vec{K}^{=}_{(i)})^2 u^{\#}_a\ ,$$

$$u^I_{a(i)} = u^I_a + \frac{1}{2} K^{=I}_{(i)} u^{\#}_a\ , \qquad u^{\#}_{a(i)} = u^{\#}_a\ .$$

Outline

8 BCFW for amplitudes from super-BCFW for superamplitudes

9 Convenient gauge with respect to $\prod_i H_i$ symmetry

10 3-point tree amplitudes with two fermionic particles and 4-fermion tree amplitude in 10D SYM

Explicit expression for the tree 4–fermion amplitudes of 10D SYM from BCFW relation

As an example we can discuss the expression for 3-point 10D SYM amplitude with two fermionic outcomes:

$$\mathcal{A}_{\dot{q}_1\ \dot{q}_2\ I}(\rho^{\#}_{(1)}, \hat{v}^{-}_{q(1)}; \rho^{\#}_{(2)}, v^{-}_{q(2)}; \rho^{\#}_{(12)}(z_{12}), v^{-}_{q}(z_{12})) =$$
$$\propto \sqrt{|\rho^{\#}_{(1)}\rho^{\#}_{(2)}|}\left[\widehat{v^{-\alpha}_{\dot{q}_1(1)}} v^{-}_{\alpha p_2(2)}\gamma^{J}_{p_2\dot{q}_2}(u^{J}_{(2)}u^{I}_{(3)}) - v^{-\alpha}_{\dot{q}_2(2)}\widehat{v^{-}_{\alpha p_1(1)}}\gamma^{J}_{p_1\dot{q}_1}(u^{J}_{(1)}u^{I}_{(3)})\right]$$

In the above gauge this simplifies to

$$\mathcal{A}_{\dot{q}_1\ \dot{q}_2\ I}(\rho^{\#}_{(1)}, \hat{v}^{-}_{q(1)}; \rho^{\#}_{(2)}, v^{-}_{q(2)}; \rho^{\#}_{(12)}(z_{12}), v^{-}_{q}(z_{12})) = \sqrt{|\rho^{\#}_{(1)}\rho^{\#}_{(2)}|}\,\widehat{K^{=I}_{(12)}}\delta_{\dot{q}_1\dot{q}_2}\ .$$

Now we can easily calculate the tree 4–fermion amplitudes of 10D SYM from BCFW relation

$$\mathcal{A}_{\dot{q}_1\dot{q}_2\dot{q}_3\dot{q}_4}(k_1; k_2; k_3; k_4) =$$
$$= \quad \propto \sqrt{|\rho^{\#}_{(1)}\rho^{\#}_{(2)}\rho^{\#}_{(3)}\rho^{\#}_{(4)}|}\left(\delta_{\dot{q}_1\dot{q}_2}\delta_{\dot{q}_3\dot{q}_4}\frac{\widehat{K^{=I}_{(12)}}\widehat{K^{=I}_{(34)}}}{(\overrightarrow{K}^{=I}_{(12)})^2} - \delta_{\dot{q}_1\dot{q}_3}\delta_{\dot{q}_2\dot{q}_4}\frac{\widehat{K^{=I}_{(13)}}\widehat{K^{=I}_{(24)}}}{(\overrightarrow{K}^{=I}_{(13)})^2}\right)\ .$$

We can write also the analogous 11D superamplitudes but these are more complicated.