Ginzburg Centennial Conference on Physics
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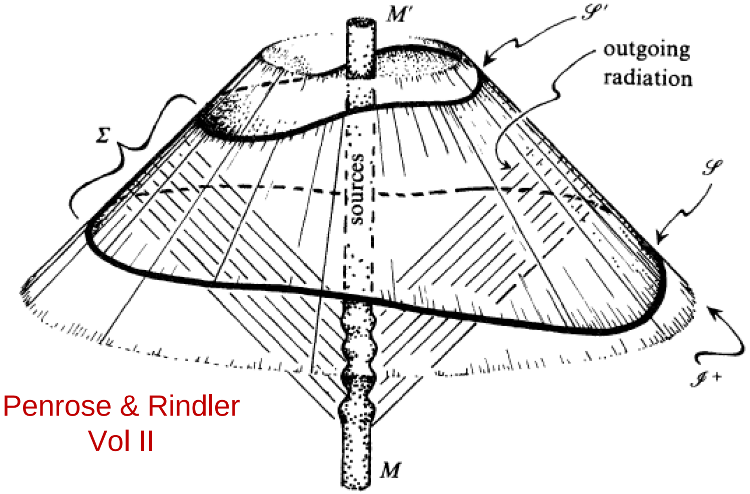
BMS current algebra and central extension



Penrose & Rindler
Vol II

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Overview

BMS symmetry

Would-be conserved BMS current algebra

The field dependent central charge

Cardyology at null infinity

In collaboration
with C. Troessaert

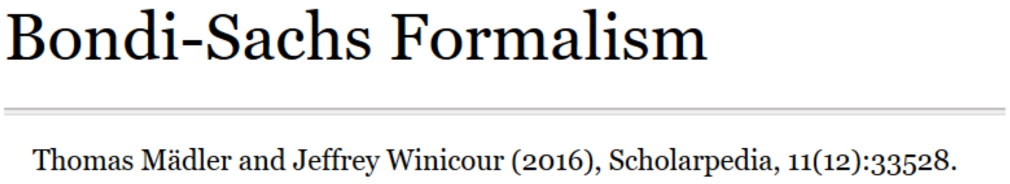
Introduction

Bondi mass loss due to gravitational radiation :

non-linear GR effect that was important to settle the controversy on the existence of gravitational waves

D. Kennefick

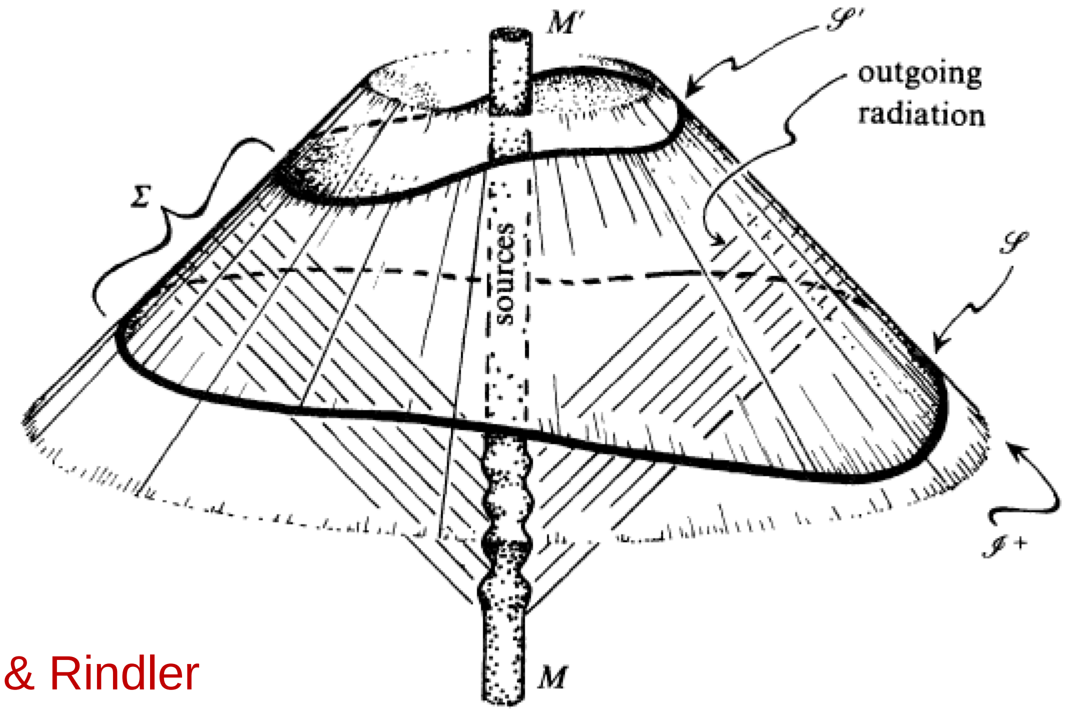
King's College and the story of how gravitational waves became real



Bondi-Sachs Formalism

Thomas Mädler and Jeffrey Winicour (2016), Scholarpedia, 11(12):33528.

The set-up



u

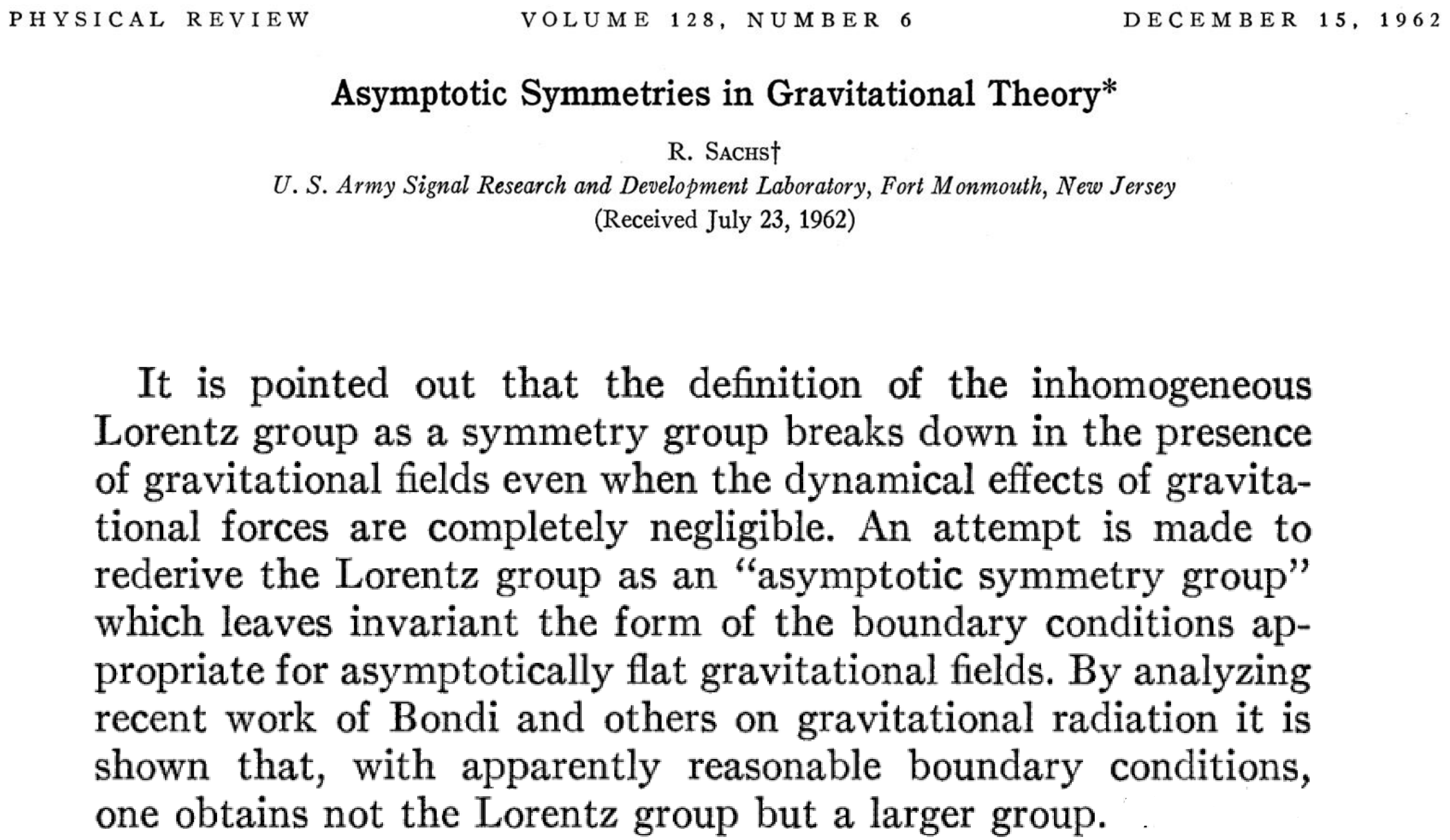
retarded time

Penrose & Rindler
Vol II

complex coordinates on
celestial sphere

$$\left[\begin{aligned} &\zeta = e^{i\phi}\cot\frac{\theta}{2} \\ &d\theta^2 + \sin^2\theta d\phi^2 = 2P_S{}^{-2}d\zeta d\bar{\zeta} \\ &P_S(\zeta,\bar{\zeta}) = \frac{1}{\sqrt{2}}(1+\zeta\bar{\zeta}) \end{aligned}\right.$$

The paper



PHYSICAL REVIEW VOLUME 128, NUMBER 6 DECEMBER 15, 1962

Asymptotic Symmetries in Gravitational Theory*

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(Received July 23, 1962)

It is pointed out that the definition of the inhomogeneous Lorentz group as a symmetry group breaks down in the presence of gravitational fields even when the dynamical effects of gravitational forces are completely negligible. An attempt is made to rederive the Lorentz group as an "asymptotic symmetry group" which leaves invariant the form of the boundary conditions appropriate for asymptotically flat gravitational fields. By analyzing recent work of Bondi and others on gravitational radiation it is shown that, with apparently reasonable boundary conditions, one obtains not the Lorentz group but a larger group.

The algebra

Poincaré algebra

GR choice: globally well-defined quantities

CFT choice: allow for poles

$$sl(2,\mathbb{C}) \ltimes ST$$

Lorentz generators as globally well-defined conformal Killing vectors fields of celestial sphere

$$[l_m, l_n] = (m-n) l_{m+n}$$
$$[\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n}$$
$$[l_m, \bar{l}_n] = 0$$

$$l_n = -\zeta^{n+1} \frac{\partial}{\partial \zeta}$$

superrotations

$$[l_m, P_{k,l}] = (\frac{1}{2}m - k) P_{m+k,l}$$
$$[\bar{l}_m, P_{k,l}] = (\frac{1}{2}m - l) P_{k,m+l}$$
$$[P_{kl}, P_{op}] = 0$$

$$P_{k,l} = P_S^{-1} \zeta^{k+\frac{1}{2}} \bar{\zeta}^{l+\frac{1}{2}}$$

supertranslations

Poincaré subalgebra $l_{-1}, l_0, l_1, \quad \bar{l}_{-1}, \bar{l}_0, \bar{l}_1, \quad P_{-\frac{1}{2},-\frac{1}{2}}, P_{\frac{1}{2},-\frac{1}{2}}, P_{-\frac{1}{2},\frac{1}{2}}, P_{\frac{1}{2},\frac{1}{2}}$

# BMS current algebra		Currents

$$J^u_\xi = -\frac{1}{8\pi G P_S^2}\Big[\Big(f\,(\underbrace{\Psi_2^0 + \sigma^0\dot{\bar\sigma}^0}_{\text{mass aspect}}) + \mathcal{Y}(\underbrace{\Psi_1^0 + \sigma^0\eth\bar\sigma^0 + \frac{1}{2}\eth(\sigma^0\bar\sigma^0)}_{\text{angular momentum aspect}})\Big) + \text{c.c.}\Big]$$

$$\xi \quad \left\{ \begin{array}{l} \mathcal{Y} = P_S^{-1}\bar{Y}, \quad \bar{\mathcal{Y}} = \bar{P}_S^{-1} Y \\[2em] f = P_S^{-1}T + \frac{u}{2}\psi \end{array} \right.$$

conformal Killing vectors

$T(\zeta,\bar\zeta)$ supertranslation generators

$\psi = \eth\mathcal{Y} + \bar\eth\bar{\mathcal{Y}}$

Ψ_i^0 Weyl tensor

$\sigma^0, \dot\sigma^0$ asymptotic part of shear & news

information on TT polarizations of gravity waves

BMS current algebra Action on fields

$$-\delta_{\xi}\sigma^{0} = [f\partial_{u} + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\bar{\eth}\overline{\mathcal{Y}}]\sigma^{0} - \underline{\eth^{2} f}$$

$$-\delta_{\xi}\dot{\sigma}^{0} = [f\partial_{u} + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y}]\dot{\sigma}^{0} - \underline{\eth^{2}\psi}$$

$$-\delta_{\xi}\Psi^{0}{}_{2} = [f\partial_{u} + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\overline{\mathcal{Y}}]\Psi^{0}_{2} + 2\eth f\Psi^{0}_{3}$$

$$-\delta_{\xi}\Psi^{0}{}_{1} = [f\partial_{u} + \mathcal{Y}\eth + \overline{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y} + \bar{\eth}\overline{\mathcal{Y}}]\Psi^{0}_{1} + 3\eth f\Psi^{0}_{2}$$

transformations of fields involve inhomogeneous terms

Minkowski vacuum breaks BMS invariance

BMS current algebra The formula

$$\underline{-\delta_{\xi_2} J^a_{\xi_1}} + \theta^a_{\xi_2}(-\delta_{\xi_1}\chi) = \underline{J^a_{[\xi_1,\xi_2]}} + K^a_{\xi_1,\xi_2} + \underline{\partial_b L^{[ab]}_{\xi_1\xi_2}}$$

$$x^a = (u, \zeta, \bar{\zeta})$$

local formula works with poles

breaking due to news

$$\theta^u_\xi(\delta\chi) = \frac{1}{8\pi G P_S^2}\left[f \dot{\bar{\sigma}}^0 \delta\sigma^0 + \text{c.c.} \right]$$

field dependent
central extension

$$K^u_{\xi_1,\xi_2} = \frac{1}{8\pi G P_S^2}\left[\left(\frac{1}{2} \bar{\sigma}^0 f_1 \eth^2 \psi_2 - (1 \leftrightarrow 2) \right) + \text{c.c.} \right]$$

vanishes when there are no
poles/superrotations

BMS current algebra Non-conservation

current non-conservation for $\xi_1 = \xi, \xi_2 = \partial_u$

$$\partial_u \mathcal{J}^u_\xi + \eth \mathcal{J}_\xi + \bar{\eth} \overline{\mathcal{J}_\xi} \approx \Theta^u_{\partial_u}(\delta_\xi \chi) + \mathcal{K}^u_{\xi,\partial_u}$$

charges when there
are no poles

$$Q_\xi = \oint_{S^2} d^2\Omega \, \mathcal{J}^u_\xi$$

$$\frac{d}{du} Q_\xi = \frac{1}{8\pi G} \oint_{S^2} d^2\Omega \, [\dot{\bar{\sigma}}^0 \delta_\xi \sigma^0 + \text{c.c.}]$$

Bondi mass loss formula for $\xi = \partial_u$

$$\frac{d}{du} Q_{\partial_u} = -\frac{1}{8\pi G} \oint_{S^2} d^2\Omega \, [\underbrace{\dot{\bar{\sigma}}^0 \dot{\sigma}^0 + \text{c.c.}}_{\geq 0}]$$

Central extension WZ consistency condition

in QFT the Adler-Bardeen anomaly satisfies
the Wess-Zumino consistency condition

$$\gamma A^a_\mu = D_\mu C^a \qquad \gamma C^a = -\frac{1}{2} C^b C^c f^a_{bc}$$

$$\left\{ \begin{array}{l} \mathrm{Tr}\, F^3 = d_H \omega^{0,5} \\ \gamma\omega^{0,5} + d_H \omega^{1,4} = 0 \\ \boxed{\gamma\omega^{1,4} + d_H \omega^{2,3} = 0} \\ \quad\vdots \\ \gamma\omega^{4,1} + d_H \mathrm{Tr}\, C^5 = 0 \qquad \vdots \\ \gamma \mathrm{Tr}\, C^5 = 0 \end{array} \right.$$

fermionic generators $Y \to \eta(\zeta) \qquad T \to C(\zeta, \bar{\zeta})$

$$\omega^{2,2} = d\zeta d\bar{\zeta} K^u - dud\bar{\zeta} K + dud\zeta \bar{K}$$

$$\left\{ \begin{array}{l} \boxed{\gamma\omega^{2,2} + d_H \omega^{3,1} = 0} \\ \gamma\omega^{3,1} + d_H \omega^{4,0} = 0 \\ \gamma\omega^{4,0} = 0 \end{array} \right.$$

Central extension   Extended algebroid

structure functions

$$[e_\alpha, e_\beta] = f_{\alpha\beta}^{\gamma}(\phi) e_\gamma \qquad R^i_{[\gamma}\partial_i f^{\epsilon}_{\alpha\beta]} = f^{\epsilon}_{\delta[\gamma} f^{\delta}_{\alpha\beta]}$$

$$[e_\alpha, f(\phi)] = R^i_\alpha(\phi)\partial_i f \qquad 2R^i_{[\alpha}\partial_i R^j_{\beta]} = f^{\gamma}_{\alpha\beta} R^j_\gamma$$

Lie algebra over functions

$$\xi = f^{\alpha}(\phi) e_\alpha \qquad [\xi_1, \xi_2] = (\xi_1^\alpha \xi_2^\beta f^{\gamma}_{\alpha\beta} + \delta_{\xi_1}\xi_2^\gamma - \delta_{\xi_2}\xi_1^\gamma) e_\gamma$$

$$\gamma\omega^2 = 0 \Leftrightarrow R^i_{[\gamma}\partial_i \omega_{\alpha\beta]} = \omega_{\delta[\gamma} f^{\delta}_{\alpha\beta]}$$

extended algebroid

$$[e_\alpha, e_\beta] = f^{\gamma}_{\alpha\beta}(\phi) e_\gamma + \omega_{\alpha\beta}(\phi) Z$$

needs all spatial boundary terms to vanish

Central extension Getting numbers

$$P_R = 1 \qquad K_{\xi_1,\xi_2} = \int d\zeta \int d\bar{\zeta}\left[\left(\sigma^0 f_1 \partial^3 Y_2 - (1 \leftrightarrow 2)\right) + \text{c.c.}\right]$$

from the conformal dimensions : $\partial_u^n \sigma(u,\zeta,\bar{\zeta}) = \sum_{k,l} (\partial_u^n \sigma)_{k,l}(u)\,\zeta^{-k-\frac{n-1}{2}}\,\bar{\zeta}^{-l-\frac{n+3}{2}}$

admit Laurent series (delta function singularities), integrals as residues

$$\left\{\begin{aligned}
K_{l_m,l_n} &= \frac{1}{2}u(m+1)(n+1)\sigma^0{}_{m+n-\frac{1}{2},-\frac{1}{2}}[n(n-1)-m(m-1)] \\
K_{l_m,\bar{l}_n} &= -\frac{1}{2}u(m+1)(n+1)[\sigma^0{}_{m-\frac{1}{2},n-\frac{1}{2}}m(m-1)-\bar{\sigma}^0{}_{m-\frac{1}{2},n-\frac{1}{2}}n(n-1)] \\
K_{l_m,P_{k,l}} &= \sigma^0{}_{m+k,l}\,m(m^2-1) \\
K_{P_{k,l},P_{o,p}} &= 0
\end{aligned}\right.$$

Central extension

Cardyology at null infinity ?

But $\sigma^0 = 0$ for Kerr black hole

transform Scri to a cylinder times a line by a finite superrotation $\zeta = e^{\frac{2\pi}{L}\omega}$

$$\partial^n_{u'}\sigma'^0(u',\omega,\bar{\omega}) = (\frac{2\pi}{L})^{n+1}[(\partial^n_u\sigma^0)_{k,l}(u)e^{-\frac{2\pi}{L}k\omega}e^{-\frac{2\pi}{L}l\bar{\omega}}] + (\frac{2\pi}{L})^2\underbrace{\frac{1}{4}(\delta^0_n u' + \delta^1_n)}$$

finite shift !

thermal circle $iu \sim iu + \beta$

Work in progress ….

Technical details

Asymptotic symmetries

Conserved currents and charges

NP formalism & solution space

Finite BMS transformations

Asymptotic symmetries Gauge fixation

Main idea : asymptotic symmetries = residual gauge symmetries

BMS ansatz

$$g^{\mu\nu} = \begin{array}{c} \begin{pmatrix} 0 & -e^{-2\beta} & 0 \\ -e^{-2\beta} & -\frac{V}{r}e^{-2\beta} & -U^{B}e^{-2\beta} \\ 0 & -U^{A}e^{-2\beta} & g^{AB} \end{pmatrix} \\ \begin{array}{ccc} u & \quad r \quad & x^{A} = \phi, \theta, \chi, \ldots \end{array} \end{array}$$

null coordinate

→ fix diffeomorphism invariance in d dimensions

d-1 gauge conditions

$$g^{uu} = 0 = g^{uA}$$

determinant condition

$$det\ g_{AB} = r^{2(d-2)} det\ \bar{\gamma}_{AB}$$

$$\bar{\gamma}_{AB} dx^{A} dx^{B} = e^{2\varphi} d^{d-2}\Omega$$

conformal to metric on
unit d-2 sphere

Asymptotic symmetries Residual gauge transformations

(weak) fall-off conditions

$$\begin{cases} \beta = o(1), \quad U^A = o(1), \quad \dfrac{V}{r} = o(r^2), \\ g_{AB}dx^A dx^B = r^2 \bar{\gamma}_{AB} dx^A dx^B + o(r^2) \end{cases}$$

residual symmetries leave this class of spacetimes invariant

exact conditions

$$\begin{cases} \mathcal{L}_\xi g_{rr} = 0 \\ \mathcal{L}_\xi g_{ra} = 0 \\ g^{AB} \mathcal{L}_\xi g_{AB} = 0 \end{cases} \Rightarrow \begin{cases} \xi^u = f \\ \xi^A = y^A - \partial_B f \int_r^\infty dr' e^{2\beta} g^{AB} \\ \xi^r = -\dfrac{r}{d-2}(\bar{D}_B \xi^B - \partial_B \xi^u U^B) \end{cases}$$

fix r dependence up to integration functions $f = f(u, x^A) \qquad y^A = y^A(u, x^B)$

asymptotic conditions

$$\begin{cases} \mathcal{L}_\xi g_{ur} = o(1) \\ \mathcal{L}_\xi g_{uA} = o(r^2) \\ \mathcal{L}_\xi g_{uu} = o(r^2) \\ \mathcal{L}_\xi g_{AB} = o(r^2) \end{cases} \Rightarrow \begin{cases} y^A = Y^A \\ f = e^{\varphi}[T + \dfrac{1}{2}\int_0^u du' e^{-\varphi}\psi], \\ \psi = \bar{D}_A Y^A \end{cases}$$

fix u dependence up to integration functions $T = T(x^B) \qquad Y^A = Y^A(x^B)$

conformal Killing equation d-2 sphere

$$\mathcal{L}_Y \bar{\gamma}_{AB} = \frac{2}{d-2}\psi \bar{\gamma}_{AB}$$

Asymptotic symmetries Lie algebroids

Metric dependence of bulk asymptotic Killing vectors $\xi^{\mu} = \xi^{\mu}(x, g)$

requires modified Lie bracket

$$[\xi_1, \xi_2]^{\mu}_{M} = [\xi_1, \xi_2]^{\mu} - \delta_{\xi_1}\xi_2^{\mu} + \delta_{\xi_2}\xi_1^{\mu}$$

$$\delta_{\xi} g_{\mu\nu} = \mathcal{L}_{\xi} g_{\mu\nu}$$

leads to representation of asymptotic symmetry algebra in the bulk spacetime

Particular example of a Lie algebroid

Asymptotic symmetries Weyl transformations

Motivations to keep the conformal factor $\varphi(u,\theta,\phi)$ or $P = P(u,\zeta,\bar{\zeta})$ arbitrary in

$$\bar{\gamma}_{AB}dx^A dx^B = e^{2\varphi}d^2\Omega = 2(P\bar{P})^{-1}d\zeta d\bar{\zeta}$$

1) because one can (general solution to Einstein's equation is known for this case)

2) finite left-over ambiguity in geometric definition of asymptotic flatness through conformal compactification

3) solution space manifestly contains Robinson-Trautman waves

$$ds^2 = -2Hdu^2 - 2dudr + 2r^2P^{-2}d\zeta d\bar{\zeta}$$

➡ Inclusion of Weyl transformations Gauge symmetry of dual theory

$$\varphi \to \varphi + \omega,\ P \to P + \omega$$

replace $\psi = \bar{D}_A Y^A \to \tilde{\psi} = \psi - 2\omega$ in asymptotic Killing vectors $bms_4 \oplus \mathrm{Weyl}$

Current algebra

Newman-Penrose formalism

first order Cartan formulation

$$S[\Gamma_{abc}, e_a{}^{\mu}] = \frac{1}{16\pi G}\int d^4x\, eR \qquad \eta_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

spin coefficients

$$\Gamma^{[ab]}{}_{c} = \Gamma^{[ab]}{}_{\mu} e^{\mu}{}_{c}$$

∇	$m^a \nabla l_a$	$\frac{1}{2}(n^a \nabla l_a - \overline{m}^a \nabla m_a)$	$-\overline{m}^a \nabla n_a$
D	$\kappa = \Gamma_{311}$	$\epsilon = \frac{1}{2}(\Gamma_{211} - \Gamma_{431})$	$\pi = -\Gamma_{421}$
Δ	$\tau = \Gamma_{312}$	$\gamma = \frac{1}{2}(\Gamma_{212} - \Gamma_{432})$	$\nu = -\Gamma_{422}$
δ	$\sigma = \Gamma_{313}$	$\beta = \frac{1}{2}(\Gamma_{213} - \Gamma_{433})$	$\mu = -\Gamma_{423}$
$\overline{\delta}$	$\rho = \Gamma_{314}$	$\alpha = \frac{1}{2}(\Gamma_{214} - \Gamma_{434})$	$\lambda = -\Gamma_{424}$

covariant derivative

$$\eth \eta^s = P\overline{P}^{-s}\overline{\partial}(\overline{P}^s \eta^s), \qquad \overline{\eth}\eta^s = \overline{P}P^s \partial (P^{-s}\eta^s)$$

$$[\overline{\eth}, \eth]\eta^s = \frac{s}{2} R \eta^s$$

conformal Killing vectors

$$\mathcal{Y} = P^{-1}\overline{Y}, \quad \overline{\mathcal{Y}} = \overline{P}^{-1} Y \qquad \overline{\eth}\mathcal{Y} = 0 = \eth\overline{\mathcal{Y}}$$

transformation law

$$\delta_{\mathcal{Y},\overline{\mathcal{Y}}}\eta = \left[\mathcal{Y}\eth + \overline{\mathcal{Y}}\overline{\eth} + h\eth\mathcal{Y} + \overline{h}\,\overline{\eth}\overline{\mathcal{Y}}\right]\eta \qquad (h, \overline{h}) = (\frac{s-w}{2}, \frac{-s-w}{2})$$

Current algebra Asymptotic solution space

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asymptotic solution space
free data

$\mathcal{J}^+$

$$\chi = \begin{cases} \Psi_0(u_0, r, \zeta, \bar{\zeta}) = \Psi_0^0 r^{-5} + O(r^{-6}) \\ \Psi_1^0(u_0, \zeta, \bar{\zeta}) \\ (\Psi_2^0 + \bar{\Psi}_2^0)(u_0, \zeta, \bar{\zeta}) \\ \sigma^0(u, \zeta, \bar{\zeta}) \qquad P(u, \zeta, \bar{\zeta}) \end{cases}$$

free u dependence

evolution equations

$$(\partial_u + \gamma^0 + 5\bar{\gamma}^0)\Psi_0^0 = \eth\Psi_1^0 + 3\sigma^0\Psi_2^0$$

$$(\partial_u + 2\gamma^0 + 4\bar{\gamma}^0)\Psi_1^0 = \eth\Psi_2^0 + 2\sigma^0\Psi_3^0$$

$$(\partial_u + 3\gamma^0 + 3\bar{\gamma}^0)\Psi_2^0 = \eth\Psi_3^0 + \sigma^0\Psi_4^0$$

on-shell constraints

news tensor

$$\alpha^0 = \frac{1}{2}\bar{P}\partial \ln P \qquad \mu_0 = -\frac{R}{4} \qquad \gamma^0 = -\frac{1}{2}\partial_u \ln \bar{P} \qquad \nu^0 = \bar{\eth}(\gamma^0 + \bar{\gamma}^0) \qquad \lambda^0 = (\partial_u + 3\gamma^0 - \bar{\gamma}^0)\bar{\sigma}^0$$

$$\Psi_2^0 - \bar{\Psi}_2^0 = \bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0 + \bar{\sigma}^0\bar{\lambda}^0 - \sigma^0\lambda^0 \qquad \Psi_3^0 = -\eth\lambda^0 + \bar{\eth}\mu^0 \qquad \Psi_4^0 = \bar{\eth}\nu^0 - (\partial_u + 4\gamma^0)\lambda^0$$

Current algebra Transformation of free data

BMS & Weyl transformations

$P = P(\zeta,\bar\zeta)$ $\quad f = P^{-1}\tilde T(\zeta,\bar\zeta) + \frac{u}{2}\tilde\psi$ $\quad \tilde\psi = \eth\mathcal{Y} - \bar\eth\bar{\mathcal{Y}} - 2\tilde\omega$

$S = (Y,\bar Y,\tilde T,\tilde\omega)$

$$-\delta_S\sigma^0 = [f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\bar\eth\bar{\mathcal{Y}} - \tilde\omega]\sigma^0 - \eth^2 f\,,$$

$$-\delta_S\dot\sigma^0 = [f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + 2\eth\mathcal{Y} - 2\tilde\omega]\dot\sigma^0 - \frac{1}{2}\eth^2\tilde\psi\,, \qquad = \Big|_{P=1} -\frac{1}{2}\bar\eth^3\bar Y$$

$$-\delta_S\Psi^0_4 = [f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + \frac{1}{2}\eth\mathcal{Y} + \frac{5}{2}\bar\eth\bar{\mathcal{Y}} - 3\tilde\omega]\Psi^0_4\,,$$

$$-\delta_S\Psi^0_3 = [f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + \eth\mathcal{Y} + 2\bar\eth\bar{\mathcal{Y}} - 3\tilde\omega]\Psi^0_3 + \eth f\Psi^0_4\,,$$

$$-\delta_S\Psi^0_2 = [f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar\eth\bar{\mathcal{Y}} - 3\tilde\omega]\Psi^0_2 + 2\eth f\Psi^0_3\,,$$

$$-\delta_S\Psi^0_1 = [f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + 2\eth\mathcal{Y} + \bar\eth\bar{\mathcal{Y}} - 3\tilde\omega]\Psi^0_1 + 3\eth f\Psi^0_2\,.$$

non trivial orbit of Minkowski under BMS

(field dependent) inhomogeneous pieces, Schwarzian derivatives

Strominger: soft gravitons = Goldstone modes for these transformations

Minkowski vacuum: $\sigma^0 = 0$ breaks BMS invariance

Current algebra Motivation/Global symmetries

Interpretation requires charges, canonical generators for the transformations
+ Dirac bracket algebra

Problem: some ADM type charges diverge because of poles on the sphere

Local non integrated version of Ward identities

$$\partial^x_\mu \langle j^\mu_{Q_1}(x)\, j^\nu_{Q_2}(y)\, X(z)\rangle = i\delta(x-y)\langle j^\nu_{[Q_1,Q_2]}(y)\, X(z)\rangle + i\delta(x-z)\langle j^\nu_{Q_2}(y)\,\delta_{Q_1} X(z)\rangle$$

classical version $\delta_{Q_1} \quad : \quad d_H\, j_{Q_2} = Q^i_2 \dfrac{\delta L}{\delta \phi^i} d^n x$

$$\Longrightarrow \quad d_H(\delta_{Q_1} j_{Q_2} - j_{[Q_1,Q_2]} - T_{Q_1,Q_2}) = 0 \qquad T_{Q_1,Q_2} \approx 0$$

$$\delta_{Q_1} j_{Q_2} = j_{[Q_1,Q_2]} + T_{Q_1,Q_2} + d_H \eta^{n-2} + K_{Q_1,Q_2}$$

$T_{Q_1,Q_2} + d_H \eta^{n-2} \sim 0$

trivial Noether current,
Belinfante ambiguities

Classification

$[j] \leftrightarrow [Q]$

$R^i_\alpha(f^\alpha) + T^i \sim 0$

$T^i \approx 0$

central extension highly
constrained

$[K_{Q_1,Q_2}] \in H^{n-1}(d_H)$

may be field dependent

cocycle condition

$$\delta_{Q_1} K_{Q_2,Q_3} - \frac{1}{2} K_{[Q_1,Q_2],Q_3} + \text{cyclic}\,(1,2,3) = 0$$

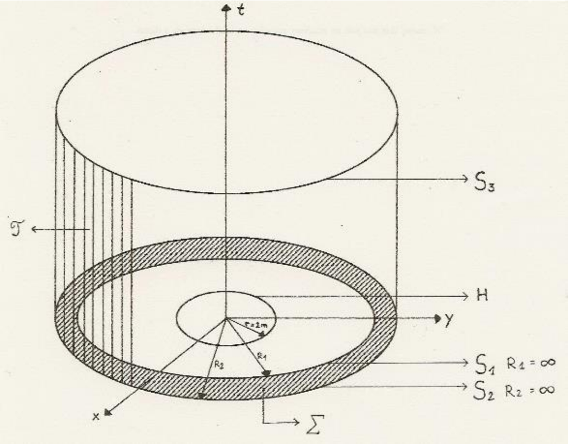
Current algebra   Gauge symmetries/Holography

gauge symmetries $\delta_f \phi^i = R^i_\alpha(f^\alpha) = R^i_\alpha f^\alpha + R^{i\mu}_\alpha \partial_\mu f^\alpha + \dots$

trivial Noether current $S_f = (R^{i\mu}_\alpha f^\alpha \frac{\delta L}{\delta \phi^i} + \dots)(d^{n-1}x)_\mu$

Classification $d_H k^{n-2} \approx 0$ $[k] \leftrightarrow [f]$ $R^i_\alpha(f^\alpha) = 0$

no solution in full GR, in linearized GR solutions classified by Kvf of background



constructive $k_f[\delta\phi] = (\frac{1}{2}\delta\phi^i \frac{\partial}{\partial\partial_\nu \phi^i} + \dots)\frac{\partial}{\partial dx^\nu} S_f$

ADM-type charges $\delta\!\!\!/ \mathcal{Q}_f[\delta\phi, \phi] = \int_{t=T, r=R} d\sigma_i \; k^{0i}_f$

conservation in time and in the bulk

asymptotic case $x^\mu = (u, r, x^A)$ $r \to \infty$

$k_f = k^{[\mu\nu]}_f (d^{n-2}x)_{\mu\nu}$ ➡ current of lower dimensional theory

$x^a = (u, x^A)$

integrability ? $k^{[ur]}_f \approx \delta J^u_f, k^{[Ar]} \approx \delta J^A_f$ conservation ?

Finite transformations BMS and Weyl group

”integrate” BMS Lie algebra → group
finite transformations of solution space

Residual gauge symmetries : find the local Lorentz transformations + diffeomorphisms that leave NPU solution space invariant
How do they act on solution space ?

$$(\zeta'(\zeta), \bar{\zeta}'(\bar{\zeta}), \beta(\zeta, \bar{\zeta}), E(u, \zeta, \bar{\zeta}) = E_R + iE_I)$$

finite superrotations, supertranslations, complex Weyl rescalings

β, E_R determine $u' = u'(u, \zeta, \bar{\zeta})$

$$\beta(\zeta, \bar{\zeta}) = \int_{\hat{u}}^{0} dv (P\bar{P})^{\frac{1}{2}}$$

$$u'(u, \zeta, \bar{\zeta}) = \int_{\hat{u}}^{u} dv e^{E_R}$$

Weyl invariant time coordinate

$$\tilde{u}(u, \zeta, \bar{\zeta}) = \int_{0}^{u} dv (P\bar{P})^{\frac{1}{2}} (v, \zeta, \bar{\zeta})$$

$$\tilde{u}'(u', \zeta', \bar{\zeta}') = J^{-\frac{1}{2}} \left[\tilde{u}(u, \zeta, \bar{\zeta}) + \beta(\zeta, \bar{\zeta}) \right]$$

$$P'(u', \zeta', \bar{\zeta}') = P(u, \zeta, \bar{\zeta}) e^{-\bar{E}} \frac{\partial \bar{\zeta}'}{\partial \bar{\zeta}}$$

$$J = \frac{\partial \zeta}{\partial \zeta'} \frac{\partial \bar{\zeta}}{\partial \bar{\zeta}'}$$

NB: simple formulas when $\partial_u P = 0 = \partial_{u'} P'$ standard BMS group when P is fixed

Finite transformations Action on solution space

$$
\begin{aligned}
\sigma'_0 &= e^{-E_R+2iE_I}\Big[\sigma_0 + \eth(e^{-E_R}\eth u') - (e^{-E_R}\eth u')(\partial_u + \overline{\gamma}^0 - \gamma^0)(e^{-E_R}\eth u')\Big],\\
\lambda'^0 &= e^{-2E}\Big[\lambda^0 + (\partial_u + 3\gamma^0 - \overline{\gamma}^0)\big[\overline{\eth}(e^{-E_R}\overline{\eth} u') - (e^{-E_R}\overline{\eth} u')(\partial_u + \gamma^0 - \overline{\gamma}^0)(e^{-E_R}\overline{\eth} u')\big]\Big],\\
{\Psi'}^0_4 &= e^{-3E_R-2iE_I}\Psi^0_4,\\
{\Psi'}^0_3 &= e^{-3E_R-iE_I}\Big[\Psi^0_3 - e^{-E_R}\eth u'\Psi^0_4\Big],\\
{\Psi'_2}^0 &= e^{-3E_R}\Big[\Psi^0_2 - 2e^{-E_R}\eth u'\Psi^0_3 + (e^{-E_R}\eth u')^2\Psi^0_4\Big],\\
{\Psi'}^0_1 &= e^{-3E_R+iE_I}\Big[\Psi^0_1 - 3e^{-E_R}\eth u'\Psi^0_2 + 3(e^{-E_R}\eth u')^2\Psi^0_3 - (e^{-E_R}\eth u')^3\Psi^0_4\Big],\\
{\Psi'}^0_0 &= e^{-3E_R+2iE_I}\Big[\Psi^0_0 - 4e^{-E_R}\eth u'\Psi^0_1 + 6(e^{-E_R}\eth u')^2\Psi^0_2 - 4(e^{-E_R}\eth u')^3\Psi^0_3 + (e^{-E_R}\eth u')^4\Psi^0_4\Big].
\end{aligned}
$$

For the Riemann sphere P=1

$$
\begin{aligned}
&\gamma^0_R = 0 = \nu^0_R = \mu^0_R, \quad \lambda^0_R = \dot{\overline{\sigma}}{}^0_R,\\
&\Psi^0_{2R} - \overline{\Psi}{}^0_{2R} = \partial^2\sigma^0_R - \overline{\partial}{}^2\overline{\sigma}{}^0_R + \dot{\sigma}^0_R\overline{\sigma}{}^0_R - \dot{\overline{\sigma}}{}^0_R\sigma^0_R,\\
&\Psi^0_{3R} = -\overline{\partial}\dot{\overline{\sigma}}{}^0_R, \quad \Psi^0_{4R} = -\ddot{\overline{\sigma}}{}^0_R,
\end{aligned}
$$

$$
\begin{aligned}
\partial_u\Psi^0_{0R} &= \overline{\partial}\Psi^0_{1R} + 3\sigma^0_R\Psi^0_{2R},\\
\partial_u\Psi^0_{1R} &= \overline{\partial}\Psi^0_{2R} + 2\sigma^0_R\Psi^0_{3R},\\
\partial_u\Psi^0_{2R} &= \overline{\partial}\Psi^0_{3R} + \sigma^0_R\Psi^0_{4R},
\end{aligned}
$$

Finite transformations From the Riemann sphere to arbitrary P

Solve evolution equation in terms of integrations functions

$$\Psi^0_{aRI} = \Psi^0_{aRI}(\zeta,\bar{\zeta})$$

$$\Psi^0_{2R} = \Psi^0_{2RI} - \bar{\eth}^2\bar{\sigma}^0_R - \sigma^0_R\dot{\bar{\sigma}}^0_R + \int_0^u dv\,\dot{\sigma}^0_R\dot{\bar{\sigma}}^0_R, \quad \Psi^0_{2RI} = \bar{\Psi}^0_{2RI},$$

Bondi mass aspect

$$(-4\pi G)M_R = \Psi^0_{2R} + \sigma^0_R\dot{\bar{\sigma}}^0_R + \underline{\bar{\eth}^2\bar{\sigma}^0_R} = \Psi^0_{2RI} + \int_0^u dv\,\dot{\sigma}^0_R\dot{\bar{\sigma}}^0_R$$

apply a pure complex rescaling

$$\zeta' = \zeta, \quad u' = \int_0^u dv\, e^{E_R}, \quad P' = e^{-\bar{E}}$$

generate solution for arbitrary P from P=1

$$\begin{aligned}
\sigma^0 &= \bar{P}^{-\frac{1}{2}}P^{\frac{3}{2}}\Big[\sigma^0_R(\tilde{u}) - \bar{\eth}^2\tilde{u}\Big],\\
\lambda^0 &= \bar{P}^2\Big[\dot{\bar{\sigma}}^0_R(\tilde{u}) - \frac{1}{2}(\eth^2\ln(P\bar{P}) + \frac{1}{2}(\eth\ln(P\bar{P}))^2)\Big],\\
\Psi^0_4 &= \bar{P}^{\frac{5}{2}}P^{\frac{1}{2}}\Big[\Psi^0_{4R}(\tilde{u})\Big],\\
\Psi^0_3 &= \bar{P}^2P\Big[\Psi^0_{3R}(\tilde{u}) + \bar{\eth}\tilde{u}\Psi^0_{4R}(\tilde{u})\Big],\\
\Psi^0_2 &= \bar{P}^{\frac{3}{2}}P^{\frac{3}{2}}\Big[\Psi^0_{2R}(\tilde{u}) + 2\bar{\eth}\tilde{u}\Psi^0_{3R}(\tilde{u}) + (\bar{\eth}\tilde{u})^2\Psi^0_{4R}(\tilde{u})\Big],\\
\Psi^0_1 &= \bar{P}P^2\Big[\Psi^0_{1R}(\tilde{u}) + 3\bar{\eth}\tilde{u}\Psi^0_{2R}(\tilde{u}) + 3(\bar{\eth}\tilde{u})^2\Psi^0_{3R}(\tilde{u}) + (\bar{\eth}\tilde{u})^3\Psi^0_{4R}(\tilde{u})\Big],\\
\Psi^0_0 &= \bar{P}^{\frac{1}{2}}P^{\frac{5}{2}}\Big[\Psi^0_{0R}(\tilde{u}) + 4\bar{\eth}\tilde{u}\Psi^0_{1R}(\tilde{u}) + 6(\bar{\eth}\tilde{u})^2\Psi^0_{2R}(\tilde{u}) + 4(\bar{\eth}\tilde{u})^3\Psi^0_{3R}(\tilde{u}) + (\bar{\eth}\tilde{u})^4\Psi^0_{4R}(\tilde{u})\Big],
\end{aligned}$$

Finite transformations   Schwarzian derivatives

transformation of the Weyl invariant quantities

$$\sigma'^0_R = \Big(\frac{\partial\zeta}{\partial\zeta'}\Big)^{-\frac{1}{2}}\Big(\frac{\partial\bar\zeta}{\partial\bar\zeta'}\Big)^{\frac{3}{2}}\Big[\sigma^0_R + \bar\partial^2\beta + \frac{1}{2}\underline{\{\bar\zeta',\bar\zeta\}(\tilde u+\beta)}\Big],$$

$$\dot{\bar\sigma}'^0_R = \Big(\frac{\partial\zeta}{\partial\zeta'}\Big)^2\Big[\dot{\bar\sigma}^0_R + \frac{1}{2}\underline{\{\zeta',\zeta\}}\Big],$$

$$\Psi'^0_{4R} = \Big(\frac{\partial\zeta}{\partial\zeta'}\Big)^{\frac{5}{2}}\Big(\frac{\partial\bar\zeta}{\partial\bar\zeta'}\Big)^{\frac{1}{2}}\Psi^0_{4R},$$

$$\Psi'^0_{3R} = \Big(\frac{\partial\zeta}{\partial\zeta'}\Big)^2\frac{\partial\bar\zeta}{\partial\bar\zeta'}\Big[\Psi^0_{3R} - Y\Psi^0_{4R}\Big], \qquad Y = \bar\partial\beta + \frac{1}{2}\bar\partial\ln\frac{\partial\bar\zeta'}{\partial\bar\zeta}(\tilde u+\beta),$$

$$\Psi'^0_{2R} = \Big(\frac{\partial\zeta}{\partial\zeta'}\Big)^{\frac{3}{2}}\Big(\frac{\partial\bar\zeta}{\partial\bar\zeta'}\Big)^{\frac{3}{2}}\Big[\Psi^0_{2R} - 2Y\Psi^0_{3R} + Y^2\Psi^0_{4R}\Big],$$

$$\Psi'^0_{1R} = \frac{\partial\zeta}{\partial\zeta'}\Big(\frac{\partial\bar\zeta}{\partial\bar\zeta'}\Big)^2\Big[\Psi^0_{1R} - 3Y\Psi^0_{2R} + 3Y^2\Psi^0_{3R} - Y^3\Psi^0_{4R}\Big],$$

$$\Psi'^0_{0R} = \Big(\frac{\partial\zeta}{\partial\zeta'}\Big)^{\frac{1}{2}}\Big(\frac{\partial\bar\zeta}{\partial\bar\zeta'}\Big)^{\frac{5}{2}}\Big[\Psi^0_{0R} - 4Y\Psi^0_{1R} + 6Y^2\Psi^0_{2R} - 4Y^3\Psi^0_{3R} + Y^4\Psi^0_{4R}\Big],$$

$$(-4\pi G)M'_R = \Big(\frac{\partial\zeta}{\partial\zeta'}\frac{\partial\bar\zeta}{\partial\bar\zeta'}\Big)^{\frac{3}{2}}\Big[(-4\pi G)M_R + \bar\partial^2\partial^2\beta + \frac{1}{2}\{\bar\zeta',\bar\zeta\}(\bar\sigma^0_R + \partial^2\beta) + $$
$$+ \frac{1}{2}\{\zeta',\zeta\}(\sigma^0_R + \bar\partial^2\beta) + \frac{1}{4}\{\bar\zeta',\bar\zeta\}\{\zeta',\zeta\}(\tilde u+\beta)\Big]$$