Independence of gauge-fixing in a higher-order BV formalism

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Moscow, May 29th, 2017

arXiv:1408.5121 DOI:10.1016/j.physletb.2015.01.005

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Δ Operator

Diff. Op.

$$\Delta(\hbar, z, \partial)$$

$$\partial_A \equiv \frac{\partial}{\partial z^A}$$

$$[\partial_A, z^B] = \delta_A^B$$

Nilpotent

$$\Delta^2 = 0$$

Grassmann-odd

$$\epsilon(\Delta) = 1$$

Planck number grading

$$\mathrm{Pl}(\hbar) = 1$$

$$\mathrm{Pl}(z) = 0$$

$$\mathrm{Pl}(\partial) = \begin{cases} 0 & \text{if } \partial \text{ acts inside } \Delta \\ -1 & \text{if } \partial \text{ acts outside } \Delta \end{cases}$$

Super Additivity

$$\mathrm{Pl}(FG) \geq \mathrm{Pl}(F) + \mathrm{Pl}(G)$$

$$\mathrm{Pl}([F,G]) \geq \mathrm{Pl}(F) + \mathrm{Pl}(G) + 1$$

Triangular form

$$\mathrm{Pl}(\Delta) \geq -2$$

Higher-Order Δ Operator, triangular form

$$\Delta \;=\; \sum_{n=-2}^{\infty}\sum_{m=0}^{n+2}\left(\frac{\hbar}{i}\right)^{n}\Delta_{n,m}$$

$$\Delta_{n,m} \;=\; \Delta_{n,m}^{A_1\ldots A_m}(z)\,\vec{\partial}_{A_m}\ldots\vec{\partial}_{A_1}$$

$m=\#(\partial)$						
						*
					*	*
3				*	*	*
2			*	*	*	*
1		*	*	*	*	*
0	*	*	*	*	*	*
	−2	−1	0	1	2	3 $\quad n=\#(\hbar)$

$\Delta_{0,2}$ original standard 2nd-order BV op.

Motivation

1. Closed SFT, higher antibrackets, L_∞, ...
2.

Operator Formalism $\longleftrightarrow$ Path Int. Formalism

Example: (Oversimplified) Point particle in curved space

Op. Formalism

$$\langle x_f | \exp\left\{-\frac{i}{\hbar}\hat{H}\Delta t\right\} |x_i\rangle \;=\; \langle x_f, t_f | x_i, t_i\rangle$$

$\sim$

Path Int. Formalism

$$\int_{x(t_i)=x_i}^{x(t_f)=x_f} [dx]\,[dp] \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\left(p_A\dot{x}^A - H_{\rm cl}\right)\right\}$$

Hamiltonian: Point particle in curved space

Operator DeWitt (1957)

$$\hat{H} = \frac{\hbar^2}{2}\left(\Delta_{\sqrt{g}} - \frac{R}{4} \right) + \mathcal{O}(\hbar^3)$$

Op — Beltrami Laplacian — Scalar Curvature

$\longleftrightarrow$

Function

$$H_{\mathrm{cl}} = \frac{1}{2} p_A g^{AB} p_B$$

Fct

W-X Formalism

Path Int./Partition Fct.

$$Z_X = \int \overset{1}{d\mu} \ \overset{2}{w} \ \overset{3}{x}$$

Fields & Antifields

$$z^A = \{\phi^\alpha, \phi^*_\alpha\}$$

1. Path Int. Measure

$$d\mu = \rho[dz][d\lambda]$$

Lagr. Mult. for Gauge-Fixing

$$\lambda^\alpha$$

2. Gauge-generating QME

$$w \equiv e^{\frac{i}{\hbar}W} \qquad (\Delta w) = 0 \qquad \mathrm{Pl}(W) \geq 0$$

3. Gauge-Fixing QME

$$x \equiv e^{\frac{i}{\hbar}X} \qquad (\Delta^T x) = 0 \qquad \mathrm{Pl}(X) \geq 0$$

Transposed Operator (Int. by parts)

Transposed Operator F^T

$$\int d\mu \, (F^T f) \, g \; = \; (-1)^{fF} \int d\mu \, f \, (Fg)$$

$$(F+G)^T \; = \; F^T + G^T$$

$$(z^A)^T \; = \; z^A$$

$$(FG)^T \; = \; (-1)^{FG} G^T F^T$$

$$(\partial_A)^T \; = \; -\rho^{-1} \overrightarrow{\partial}_A \rho$$

Transposed Operator (Int. by parts)

Affine Leibniz rule

$$\partial_A^T(fg) \;=\; (\partial_A^T f)g - (-1)^{Af} f(\partial_A g)$$

Transposed Δ operator also nilpotent

$$(\Delta^T)^2 \;=\; 0$$

Original 2nd-Order BV (Int. out λ & ϕ^*)

Path int. measure density

$$\rho = 1$$

Darboux Coord.

$$\Delta = (-1)^{\alpha} \frac{\delta}{\delta\phi^{\alpha}} \frac{\delta}{\delta\phi^{*}_{\alpha}} = \Delta^{T}$$

Gauge-Fixing Action

$$X = \left(\phi^{*}_{\alpha} - \frac{\delta\psi}{\delta\phi^{\alpha}}\right)\lambda^{\alpha}$$

CME

$$(X, X) = 0$$

$\mathcal{O}(\hbar)$

$$(\Delta^{T} X) = 0$$

QME

$$(\Delta^{T} \exp\left\{\frac{i}{\hbar}X\right\}) = 0 \qquad \Leftrightarrow \qquad \frac{1}{2}(X, X) = i\hbar(\Delta^{T} X)$$

Original 2nd-Order BV (1981)

Path Int./Partition Fct.

$$Z_{\psi} \;=\; \int [d\phi] \exp\left\{\frac{i}{\hbar} W(\phi, \phi^{*} = \frac{\delta\psi}{\delta\phi}\right\})$$

QME

$$(\Delta \exp\left\{\frac{i}{\hbar} W\right\}) \;=\; 0 \qquad \Leftrightarrow \qquad \frac{1}{2}(W, W) \;=\; i\hbar(\Delta W)$$

Independence of gauge-fixing in a higher-order BV formalism

Finite Deformation of Solution to QME

$$x \quad \longrightarrow \quad x' \;=\; \left(e^{[\Delta^T,\Psi]}x\right)$$
$$=\; x + \left([\Delta^T,\Psi]x\right) + \frac{1}{2}\left([\Delta^T,\Psi][\Delta^T,\Psi]x\right) + \ldots$$

$$(\Delta^T x) \;=\; 0 \quad \longrightarrow \quad (\Delta^T x') \;=\; 0$$

Deformation generating operator

$$\Psi(\hbar, z, \partial) \qquad \mathrm{Pl}(\Psi) \;\geq\; 0 \qquad \varepsilon(\Psi) \;=\; 1$$

Infinitesimal Deformation of Solution to QME

Δ^T-closed

$$(\Delta^T \delta x) \;=\; 0$$

Δ^T-exact

$$\delta x \;=\; ([\Delta^T, \Psi] x) \;=\; (\Delta^T \Psi x)$$

Assumption

No Δ^T-cohomology in pertinent sector

Independence of Gauge-Fixing: Int. by parts

Int. by parts Batalin, Damgaard & KB (1996)

$$\delta Z \equiv Z_{X+\delta X} - Z_X = \int d\mu\ w\ \delta x$$

$$= \int d\mu\ w\ (\Delta^T \Psi x) = \int d\mu\ (\Delta w)\ (\Psi x) = 0$$

Question

Can we construct proof using change of int. var. instead?

Homotopy operator $\overset{\rightarrow A}{h}(\Delta)$

Δ operator

$$\Delta(\partial, z) = \sum_{m=0}^{\infty} \Delta_m(\partial, z)$$

Anti-normal order

$$\Delta_m(\partial, z) = \overset{\rightarrow}{\partial}_{A_m} \cdots \overset{\rightarrow}{\partial}_{A_1} \Delta_m^{A_1 \ldots A_m}(z)$$

Homotopy operator

$$\overset{\rightarrow A}{h}(\Delta_m) := \begin{cases} -\frac{1}{m}[z^A, \Delta_m] & \text{for} \quad m \geq 1 \ , \\ 0 & \text{for} \quad m = 0 \ . \end{cases}$$

$\rightsquigarrow$ Extended by linearity

Homotopy Formula

$$(-1)^A \overrightarrow{\partial}_A \overrightarrow{h}^A (\Delta(\partial, z)) = \Delta(\partial, z) - \Delta(0, z)$$

$$\overrightarrow{h}^A \overrightarrow{h}^B (\Delta) = (-1)^{AB} \overrightarrow{h}^B \overrightarrow{h}^A (\Delta)$$

Bilinear Homotopy Operator $B^A(f,\Delta)$

$$
\begin{aligned}
(-1)^{Af} B^A(f,\Delta) &= f \; : \frac{1}{1 - \overleftarrow{\partial}_B^T \; \overrightarrow{h}^B} : \; \overrightarrow{h}^A(\Delta)1 \\
&\equiv f \; : \sum_{n=0}^{\infty} \left(\overleftarrow{\partial}_B^T \; \overrightarrow{h}^B \right)^n : \; \overrightarrow{h}^A(\Delta)1 \\
&= f \; \underbrace{\overrightarrow{h}^A(\Delta)1}_{n=0} + \underbrace{(f \; \overleftarrow{\partial}_B^T) \; \overrightarrow{h}^B \overrightarrow{h}^A(\Delta)1}_{n=1} \\
&\quad + \underbrace{(f \; \overleftarrow{\partial}_B^T \overleftarrow{\partial}_C^T) \; \overrightarrow{h}^C \overrightarrow{h}^B \overrightarrow{h}^A(\Delta)1}_{n=2} + \ldots
\end{aligned}
$$

Bilinear Homotopy Formula

$$(-1)^{A}\left(\partial_A^T B^A(f,\Delta)\right) \;=\; (-1)^{f\Delta}(\Delta^T f) - f(\Delta 1)$$

Infinitesimal change of Int. Variables

$$\delta z^A = \frac{1}{wx} B^A(\Psi x, \overrightarrow{\Delta} w)$$

Ψ infinitesimal & Grassmann-odd op.

Divergence

$$\mathrm{div}_{\rho wx}\delta z = \frac{(-1)^A}{\rho wx}(\overrightarrow{\partial}_A \rho x w\, \delta z^A) = -\frac{(-1)^A}{wx}(\overrightarrow{\partial}_A^T B^A(\Psi x, \overrightarrow{\Delta} w))$$

$$= \frac{1}{wx}\left\{(\Psi x)\underbrace{(\Delta w)}_{=0} + w(\Delta^T \Psi x)\right\} = \frac{\delta x}{x}$$

Independence of Gauge-Fixing

Change of Int. Variables $z'^A \;=\; z^A + \delta z^A$

$$
\begin{aligned}
0 \;&=\; \int[d\lambda][dz']\rho[z']\; w[z']\; x[z'] - \int[d\lambda][dz]\rho[z]\; w[z]\; x[z] \\
&=\; \int[d\lambda][dz]\; (-1)^A(\vec{\partial}_A \rho x w\; \delta z^A) \;=\; \int d\mu\; wx\; \mathrm{div}_{\rho x w}\delta z \\
&=\; \int d\mu\; w\; \delta x \;=\; Z_{X+\delta X} - Z_X \;\equiv\; \delta Z
\end{aligned}
$$

Conclusions

- The formal higher-order BV path int/partition fct Z_X does not depend on the gauge-fixing condition X.
- Proof either via int. by parts or change of int. variables.
- It is possible to generalize to BRST/anti-BRST $Sp(2)$-symmetric theories.
- Recently, finite BRST transformations have been considered by Batalin, Lavrov, Tyutin & KB.

Classical 2nd-order case (from now on)

Δ Operator

$$\Delta = \underbrace{\Delta_\rho}_{\text{Odd Laplacian}} + \underbrace{V}_{\text{Odd vector field}} + \underbrace{\nu}_{\text{Odd scalar curvature}}$$

Nilpotent

$$\Delta^2 = 0$$

Transposed

$$\Delta^T = \Delta_\rho - V + \nu$$

QME

$$\frac{1}{2}(W,W) + \frac{\hbar}{i}\left(\left(\Delta_\rho + V\right) W\right) + \left(\frac{\hbar}{i}\right)^2 \nu \;=\; 0$$

$$\frac{1}{2}(X,X) + \frac{\hbar}{i}\left(\left(\Delta_\rho - V\right) X\right) + \left(\frac{\hbar}{i}\right)^2 \nu \;=\; 0$$

Q BRST

$$\sigma_W f \;=\; (W, f) + \tfrac{\hbar}{i}\left(\left(\Delta_\rho + V\right) f\right)$$

$$\sigma_X f \;=\; (X, f) + \tfrac{\hbar}{i}\left(\left(\Delta_\rho - V\right) f\right)$$

$$\sigma_W^2 \;=\; 0$$

$$\sigma_X^2 \;=\; 0$$

Infinitesimal Change of Int. Var.

$$2\,\delta z^A \;=\; \frac{i}{\hbar}\psi\,(\sigma_W z^A - \sigma_X z^A) - (\psi, z^A)$$

ψ infinitesimal & Grassmann-odd fct.

The Odd Scalar ν_{ρ}

Odd Scalar in Antisymplectic Geometry (KB 2006)

$$\nu_{\rho} := \nu_{\rho}^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24}$$

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Terms built from E and ρ

$$\nu_{\rho}^{(\mathbf{0})} \ := \ \frac{1}{\sqrt{\rho}}(\Delta_{1}\sqrt{\rho})$$

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$$\nu_\rho^{(0)} \;:=\; \frac{1}{\sqrt{\rho}}(\Delta_1\sqrt{\rho})$$

$$\nu^{(1)} \;:=\; (-1)^{\varepsilon_A}\left(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} E^{AB} \frac{\overleftarrow{\partial^r}}{\partial z^B}\right)(-1)^{\varepsilon_B}$$

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$$\nu^{(2)} \;:=\; -(-1)^{\varepsilon_B}\left(\frac{\overrightarrow{\partial^\ell}}{\partial z^A}E_{BC}\right)(z^C,(z^B,z^A))$$

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$$\nu^{(2)} \;:=\; -(-1)^{\varepsilon_B}\Big(\frac{\overrightarrow{\partial^\ell}}{\partial z^A}E_{BC}\Big)(z^C,(z^B,z^A))$$

$$=\; (-1)^{\varepsilon_A\varepsilon_D}\Big(\frac{\overrightarrow{\partial^\ell}}{\partial z^D}E^{AB}\Big)E_{BC}\Big(E^{CD}\frac{\overleftarrow{\partial^r}}{\partial z^A}\Big)$$

Odd scalar curvature

- ν_{ρ} is constructed from antisymplectic structure E^{AB} and ρ.
- ν_{ρ} transforms as a scalar.
- $\nu_{\rho} = 0$ can be viewed as a compatibility condition between E^{AB} and ρ.
- $\nu_{\rho} = -\frac{R}{8}$ is an odd scalar curvature for any tangent space connection, that is
    1. torsionfree,
    2. antisymplectic,
    3. compatible with ρ.

References

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