Anomalous scaling in turbulence with direct and/or inverse energy cascades

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Moscow, Russia

GINZBURG CENTENNIAL CONFERENCE on PHYSICS

Credits [in order of appearance]: **S. Musacchio** (CNRS-France); **F. Toschi** (TuE, The Netherlands); **E. Titi** (Weizmann Institute of Science, Israel), **F. Bonaccorso, M. Linkmann, M. Buzzicotti, G. Sahoo** (U. Tor Vergata, Italy), **A. Alexakis (**ENS, Paris, France)

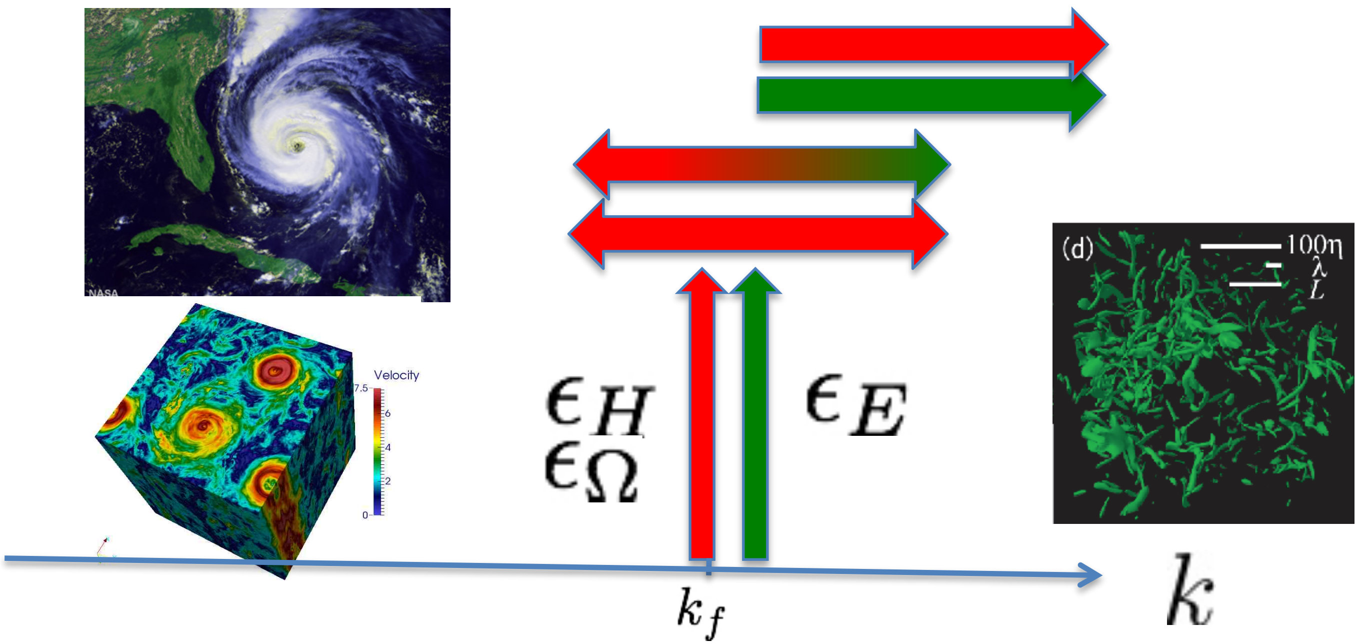
MOTIVATIONS:

A TALE ABOUT TRANSFER PROPERTIES OF INVISCID CONSERVED QUANTITIES, KINETIC ENERGY, HELICITY ENSTROPHY, MAGNETIC HELICITY ETC…

Q1: HOW TO PREDICT THE DIRECTION OF THE TRANSFER (FORWARD/BACKWARD) AND ITS ROBUSTNESS UNDER EXTERNAL PERTURBATION (FORCING/BOUNDARY CONDITIONS)?

Q2: HOW MUCH THE FLUCTUATIONS AROUND THE MEAN TRANSFER ARE INTENSE AND SELF-SIMILAR (INTERMITTENCY AND ANOMALOUS SCALING) ?

AS A MATTER OF FACT, FOR 3D NAVIER STOKES EQUATIONS, WE DO NOT KNOW HOW TO PREDICT NEITHER THE SIGN OF THE MEAN ENERGY TRANSFER NOR THE INTENSITY OF THE FLUCTUATIONS AROUND IT.



ENERGY (2D/3D)/MAGNETIC HELICITY(3D)　　ENERGY (3D)/ENSTROPHY(2D)

EXPLORING THE ROLE OF MIRROR SIMMETRY

- ROLE OF KINETIC HELICITY IN THE REVERSAL OF THE MEAN ENERGY FLUX IN 3D NAVIER-STOKES (FORWARD/BACKWARD)

- IMPLICATION FOR THE SMALL-SCALES REGULARITY OF THE NAVIER-STOKES SOLUTIONS

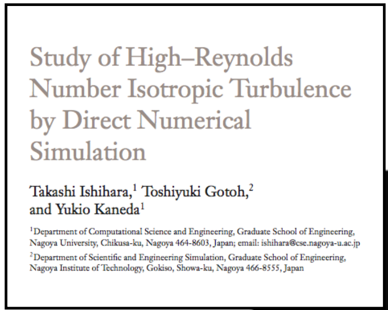
- EMPIRICAL OBSERVATION ON ROTATING TURBULENCE

- IMPLICATION FOR SMALL SCALE INTERMITTENCY AND DEVIATION FROM KOLMOGOROV 1941 SCALING

- ROLE OF MAGENTIC HELICITY IN THE FORMATION OF LARGE AND SMALL SCALES DYNAMO IN MAGNETOHYDRODYNAMICS

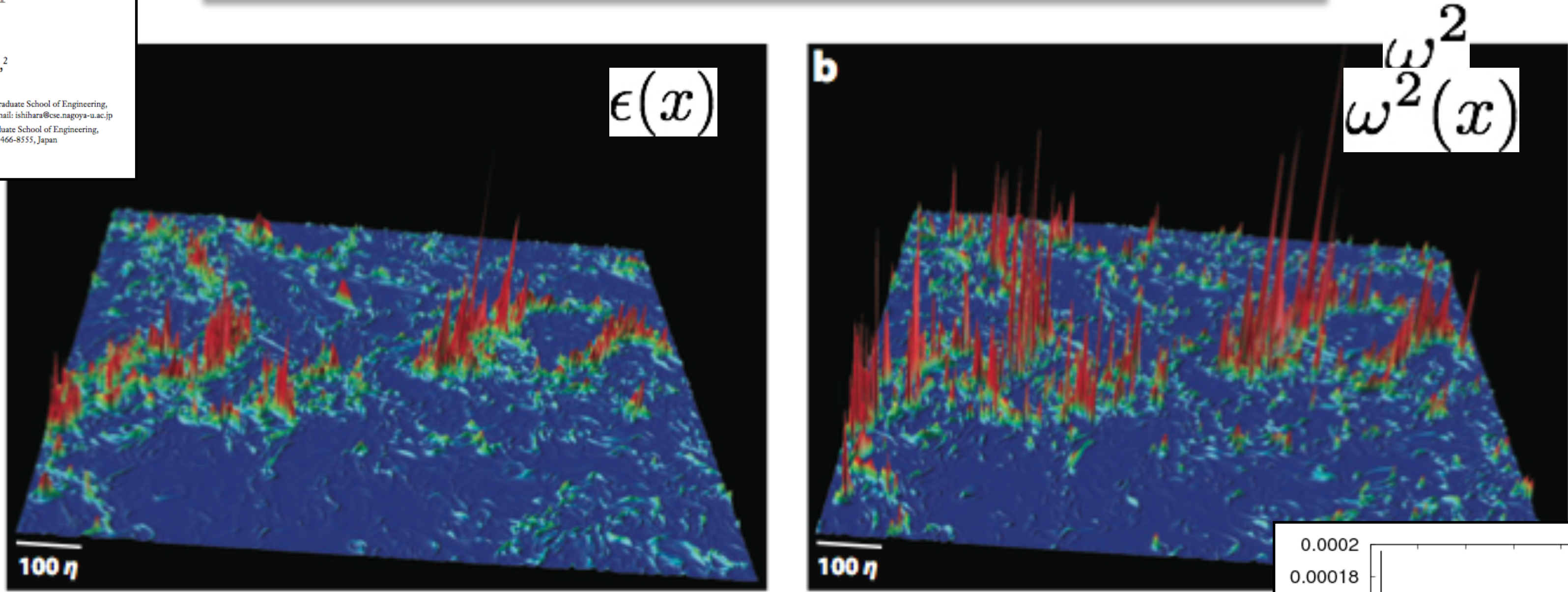
$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \partial)\mathbf{v} = -\partial P + \nu \Delta \mathbf{v} + \mathbf{F} \\ \partial \cdot \mathbf{v} = 0 \\ + \; Boundary\ Conditions \end{cases}$$

3D HOMOGENEOUS AND ISOTROPIC TURBULENCE FLUCTUATIONS: SMALL-SCALES INTERMITTENCY


Study of High–Reynolds Number Isotropic Turbulence by Direct Numerical Simulation

Takashi Ishihara,[1] Toshiyuki Gotoh,[2] and Yukio Kaneda[1]

[1]Department of Computational Science and Engineering, Graduate School of Engineering, Nagoya University, Chikusa-ku, Nagoya 464-8603, Japan; email: ishihara@cse.nagoya-u.ac.jp
[2]Department of Scientific and Engineering Simulation, Graduate School of Engineering, Nagoya Institute of Technology, Gokiso, Showa-ku, Nagoya 466-8555, Japan



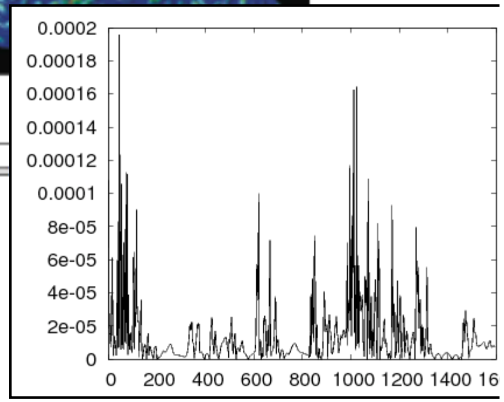
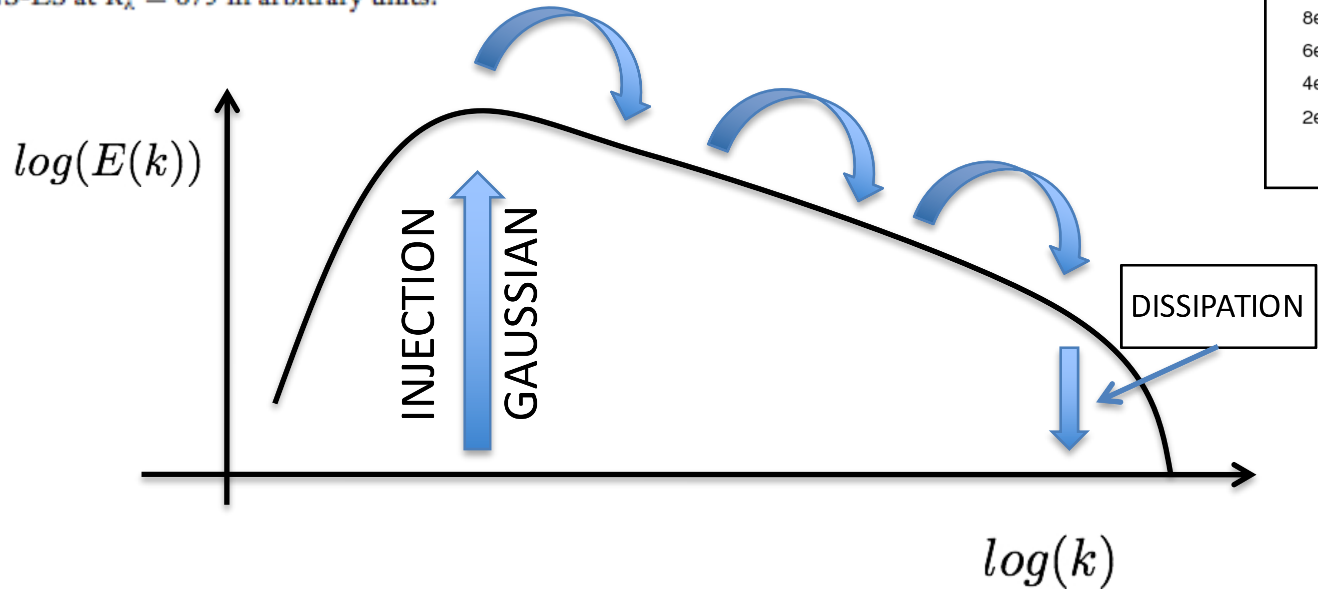


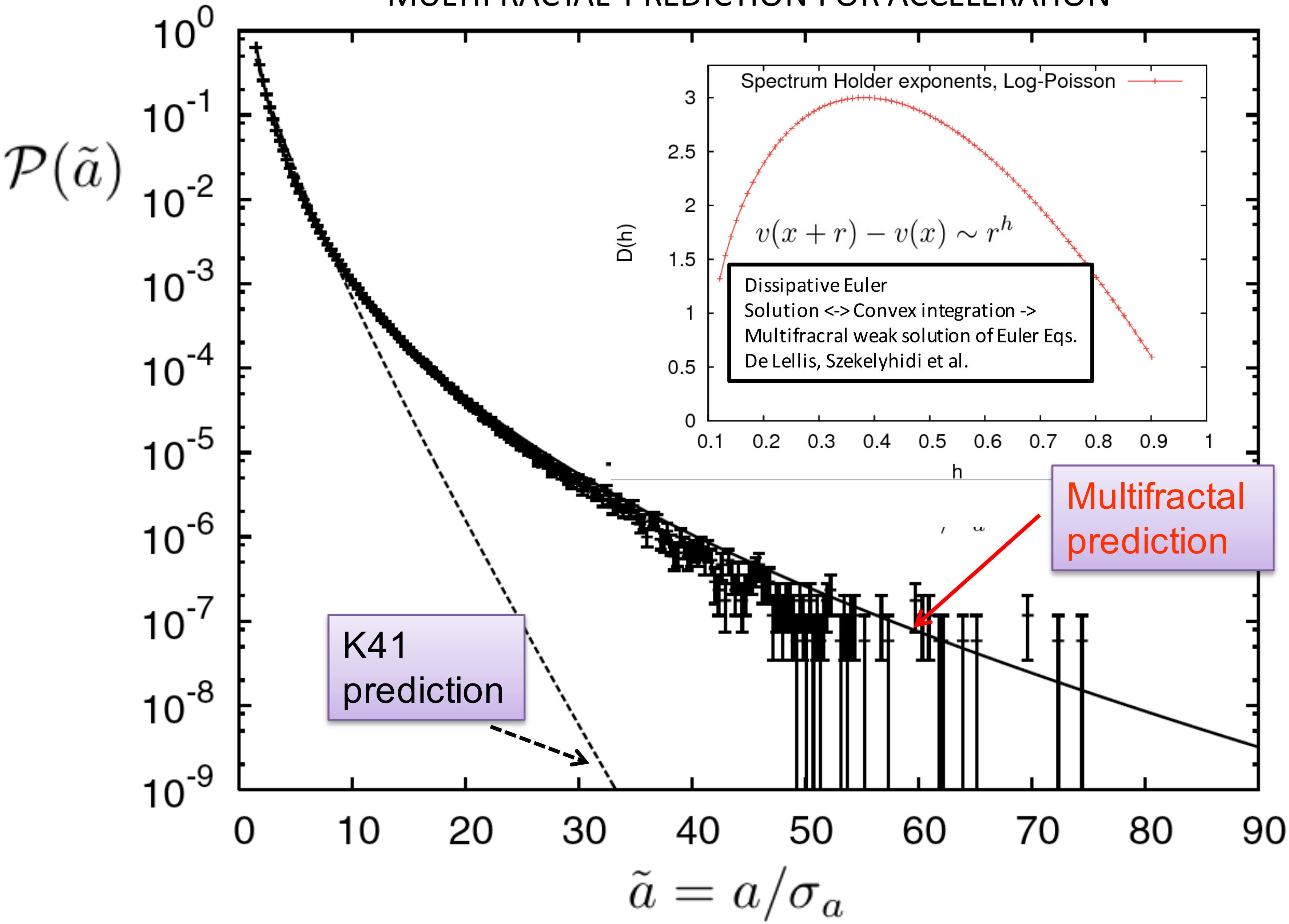
Figure 4

Snapshot of the intensity distributions of (*a*) the energy-dissipation rate $\tilde{\varepsilon} = \varepsilon/(2\nu)$ and (*b*) the enstrophy $\Omega =$ DNS-ES at $R_\lambda = 675$ in arbitrary units.



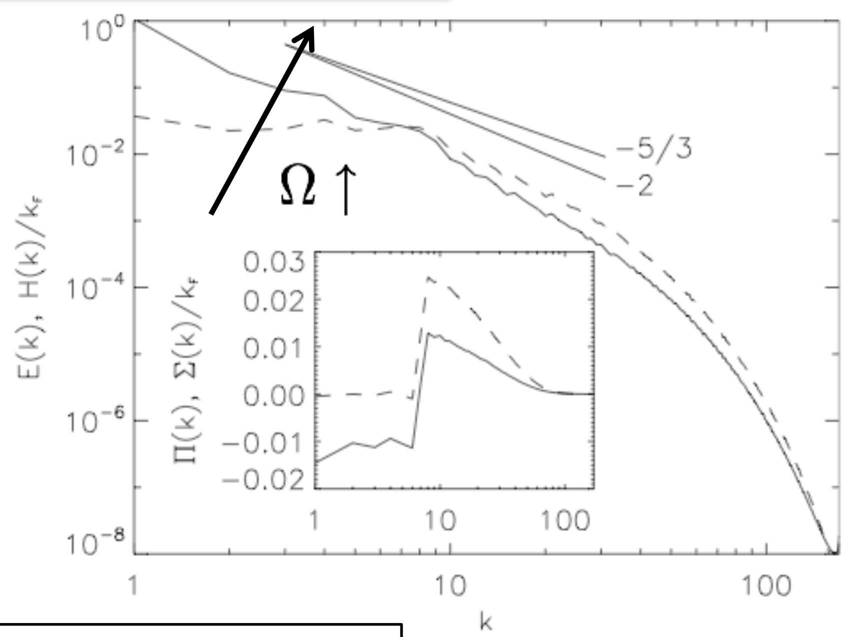
L. B., G. Boffetta, A. Celani, B. Devenish, A. Lanotte and F. Toschi *PRL* **93**, 064502, 2004

MULTIFRACTAL PREDICTION FOR ACCELERATION

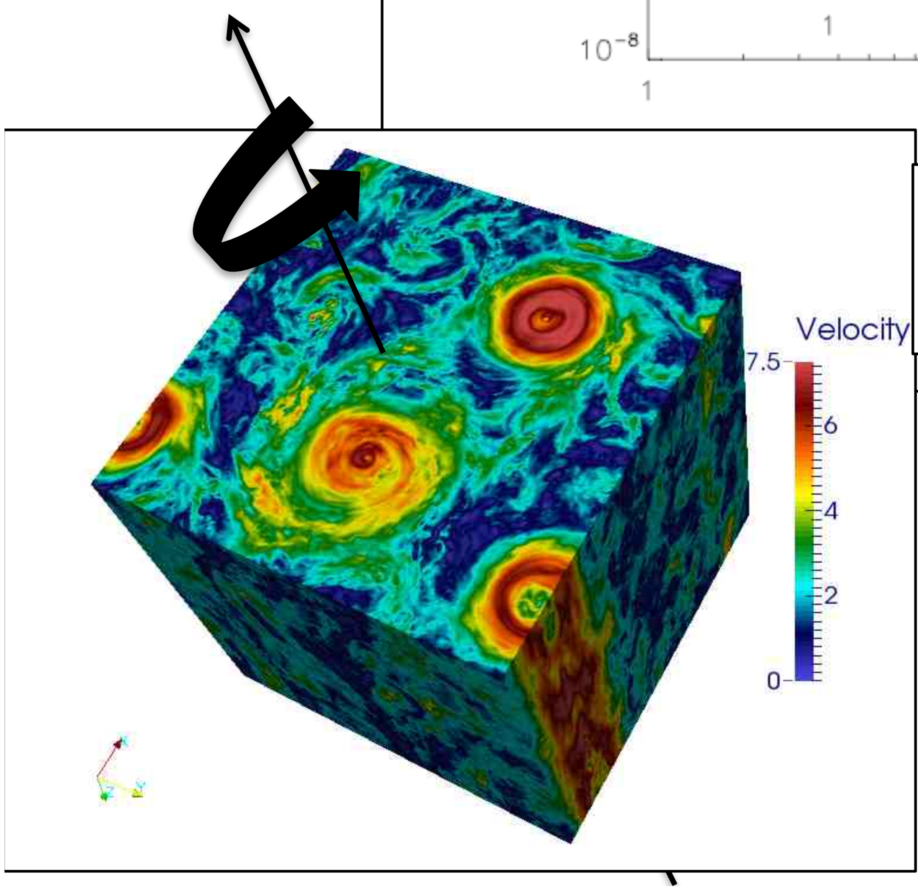


$$\mathcal{P}(a) \sim \int_{h\in I} dh\, a^{\frac{h-5+D(h)}{3}} \nu^{\frac{7-2h-2D(h)}{3}} L_0^{D(h)+h-3} \sigma_v^{-1} \times \quad \exp\left(-\frac{a^{\frac{2(1+h)}{3}} \nu^{\frac{2(1-2h)}{3}} L_0^{2h}}{2\sigma_v^2}\right)$$

P.D. Mininni and A. Pouquet.
Phys. Rev. E **79,** 026304 (2009)



INVERSE ENERGY CASCADE
UNDER ROTATION
3D -> 2D
HELICITY ENHANCEMENTS



doi: 10.1209/0295-5075/100/44003

Dual non-Kolmogorov cascades in a von Kármán flow

E. Herbert[1], F. Daviaud[1], B. Dubrulle[1], S. Nazarenko[2] and A. Naso[3]

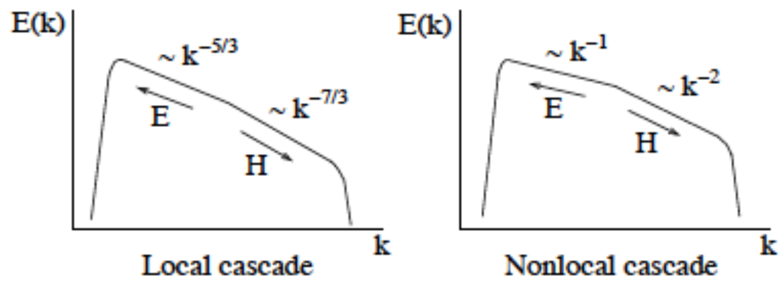
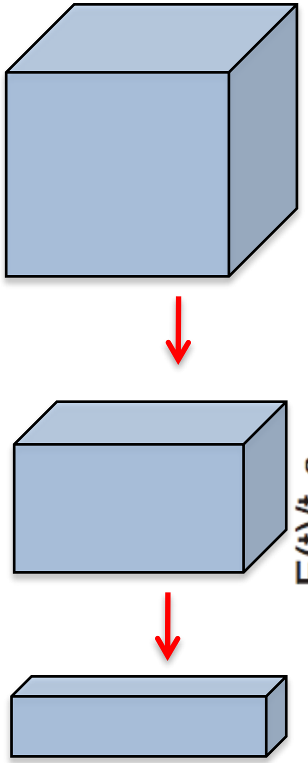


FIG. 1: Summary of the two different possibilities for dual helicity and energy cascades as a function of the wavenumber k in a Beltrami flow. Left : local case; right : non-local case.

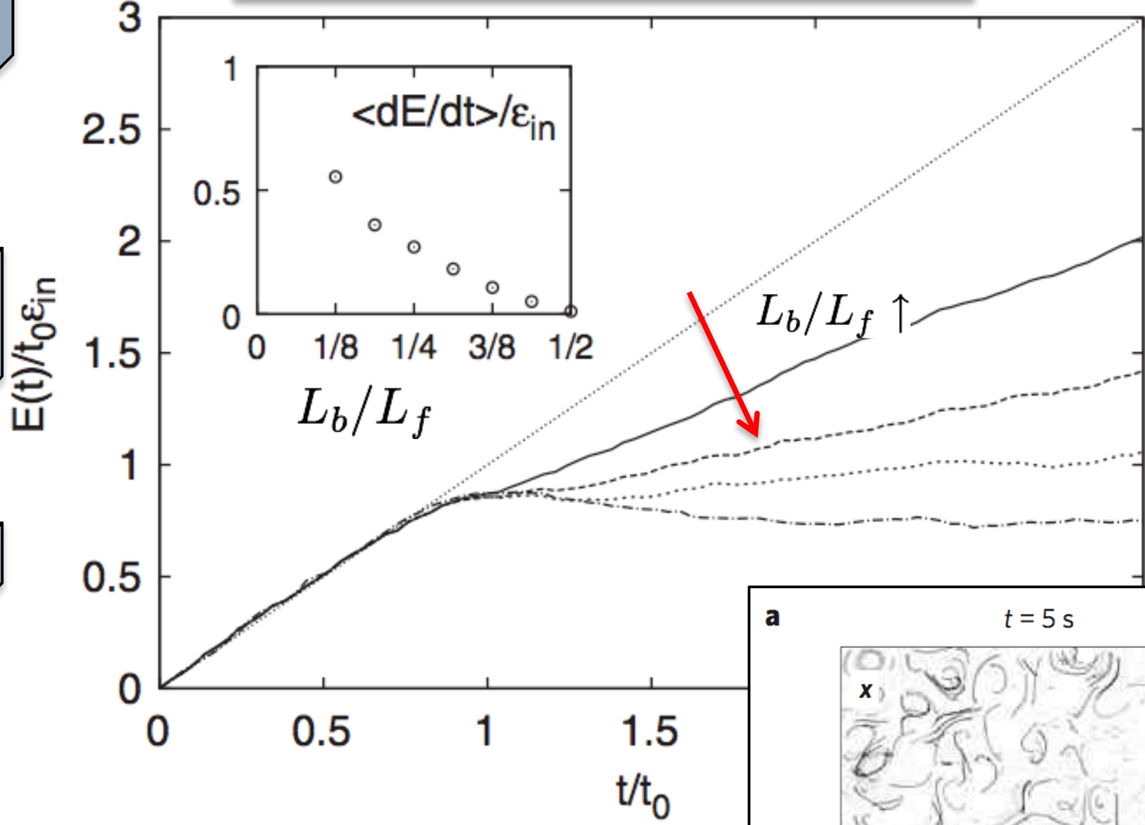
PRL 104, 184506 (2010) PHYSICAL REVIEW LETTERS week ending 7 MAY 2010

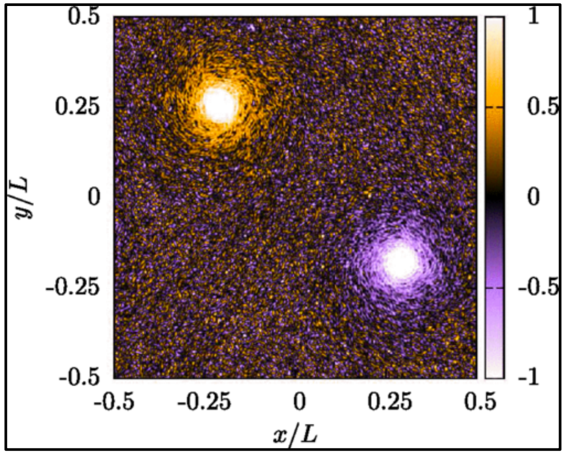
Turbulence in More than Two and Less than Three Dimensions

Antonio Celani,[1] Stefano Musacchio,[2,3] and Dario Vincenzi[3]

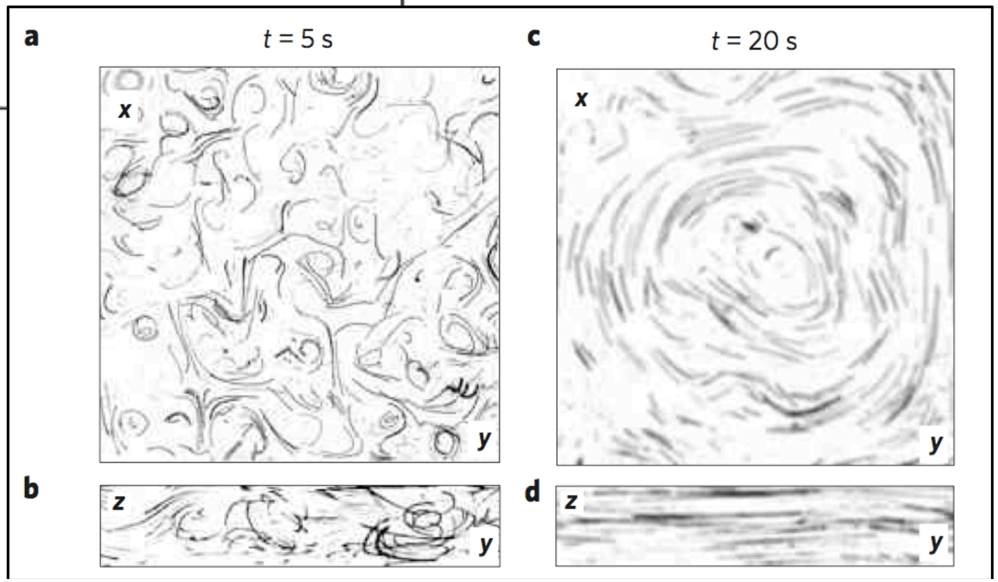


CONFINEMENT 3D → 2D





J. Laurie, G. Boffetta, G. Falkovich, I. Kolokolov, and V. Lebedev PRL **113**, 254503–2014



Upscale energy transfer in thick turbulent fluid layers

H. Xia[1], D. Byrne[1], G. Falkovich[2] and M. Shats[1]*

Q: CAN WE DISSECT 3D NS EQUATIONS TO EXTRACT INTERESTING INFORMATION FROM ITS ELEMENTARY CONSTITUENTS?

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \partial)\mathbf{v} = -\partial P + \nu \Delta \mathbf{v} + \mathbf{F} \\ \partial \cdot \mathbf{v} = 0 \\ + \; Boundary \; Conditions \end{cases}$$

Commun. Math. Phys. 115, 435–456 (1988)

The Beltrami Spectrum for Incompressible Fluid Flows

Peter Constantin[1,⋆] and Andrew Majda[2,⋆⋆]

The nature of triad interactions in homogeneous turbulence

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(Received 24 July 1991; accepted 22 October 1991)

$$\boldsymbol{u}(\boldsymbol{k}) = u^+(\boldsymbol{k})\boldsymbol{h}^+(\boldsymbol{k}) + u^-(\boldsymbol{k})\boldsymbol{h}^-(\boldsymbol{k})$$

$$\boldsymbol{h}^\pm = \hat{\boldsymbol{\nu}} \times \hat{\boldsymbol{k}} \pm i\hat{\boldsymbol{\nu}}$$

$$\hat{\boldsymbol{\nu}} = \boldsymbol{z} \times \boldsymbol{k} / ||\boldsymbol{z} \times \boldsymbol{k}||$$

$$i\boldsymbol{k} \times \boldsymbol{h}^\pm = \pm k\boldsymbol{h}^\pm$$

$$\begin{cases} E = \sum_{\boldsymbol{k}} |u^+(\boldsymbol{k})|^2 + |u^-(\boldsymbol{k})|^2; \\ H = \sum_{\boldsymbol{k}} k(|u^+(\boldsymbol{k})|^2 - |u^-(\boldsymbol{k})|^2). \end{cases}$$

$$u^{s_k}(\mathbf{k},t) \quad (s_k = \pm 1)$$

$$\frac{d}{dt}u^{s_k}(\mathbf{k}) + \nu k^2 u^{s_k}(\mathbf{k}) = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \sum_{s_p, s_q} g_{\mathbf{k},\mathbf{p},\mathbf{q}}(s_p p - s_q q) \times [u^{s_p}(\mathbf{p}) u^{s_q}(\mathbf{q})]^*. \tag{15}$$

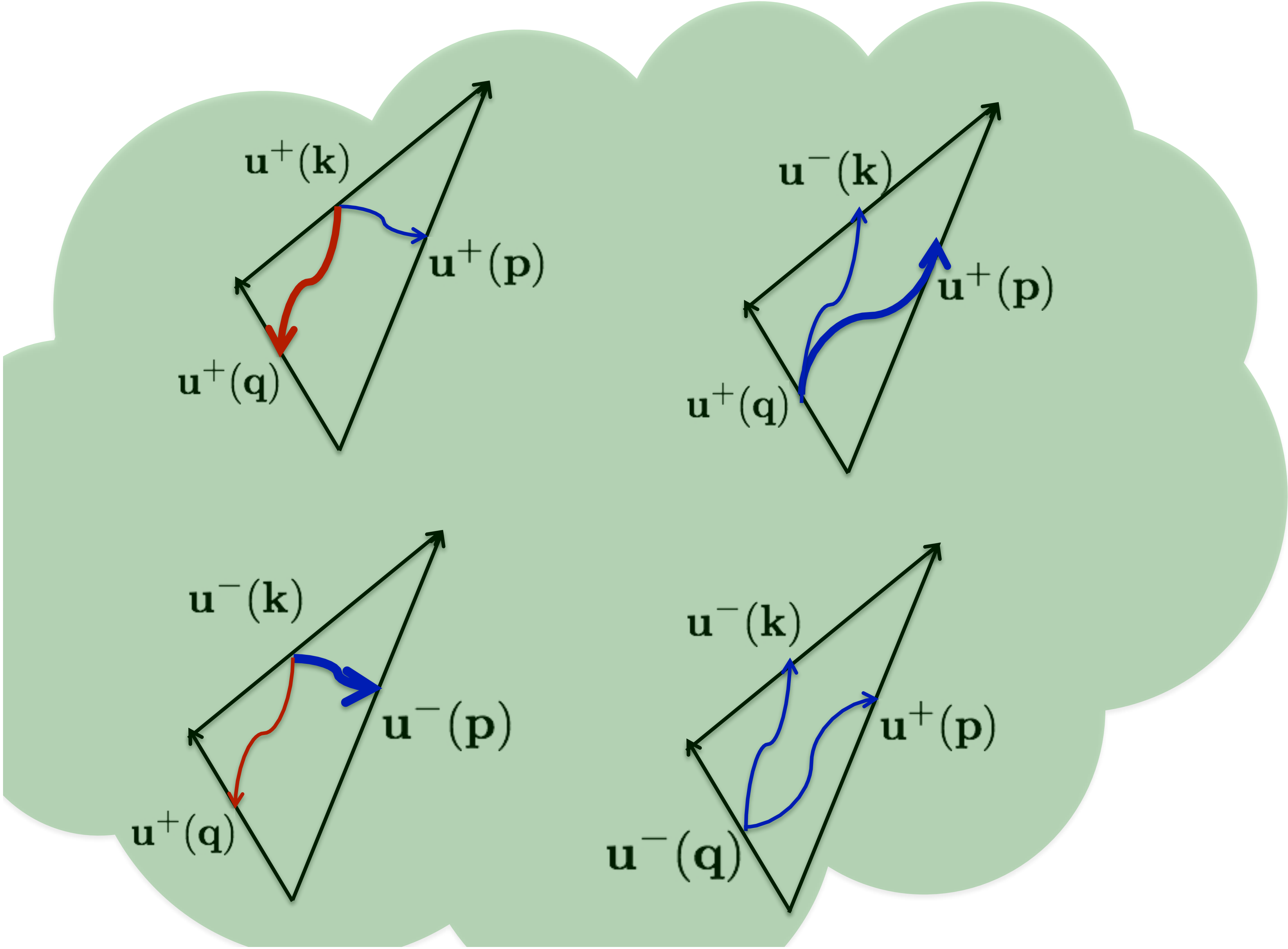
Eight different types of interaction between three modes $u^{s_k}(\mathbf{k})$, $u^{s_p}(\mathbf{p})$, and $u^{s_q}(\mathbf{q})$ with $|\mathbf{k}|<|\mathbf{p}|<|\mathbf{q}|$ are allowed according to the value of the triplet (s_k, s_p, s_q)

$$\dot{u}^{s_k} = r(s_p p - s_q q)\frac{s_k k + s_p p + s_q q}{p}(u^{s_p} u^{s_q})^*,$$

$$\dot{u}^{s_p} = r(s_q q - s_k k)\frac{s_k k + s_p p + s_q q}{p}(u^{s_q} u^{s_k})^*,$$

$$\dot{u}^{s_q} = r(s_k k - s_p p)\frac{s_k k + s_p p + s_q q}{p}(u^{s_k} u^{s_p})^*.$$

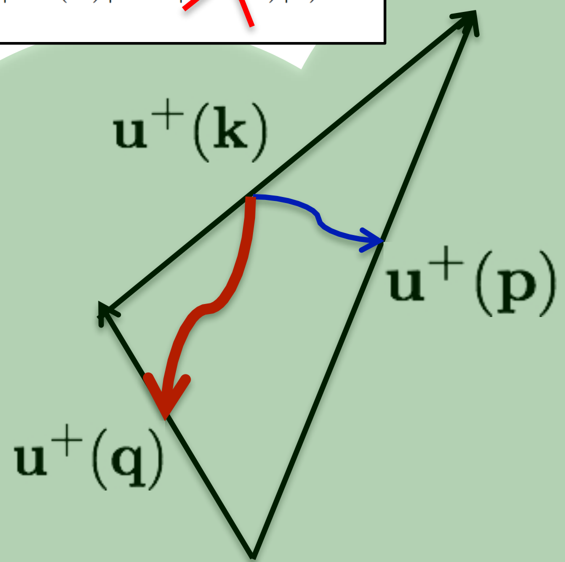
HELICAL TRIADIC INTERACTION IN THE NAVIER_STOKES EQS



TRIADIC INTERACTION IN DECIMATED NAVIER_STOKES EQS

HOMOCHIRAL

$$\begin{cases} E = \sum_{\boldsymbol{k}} |u^+(\boldsymbol{k})|^2 + \cancel{|u^-(\boldsymbol{k})|^2}; \\ H = \sum_{\boldsymbol{k}} k(|u^+(\boldsymbol{k})|^2 - \cancel{|u^-(\boldsymbol{k})|^2}). \end{cases}$$



HOMOCHIRAL 3D NAVIER STOKES EQS.

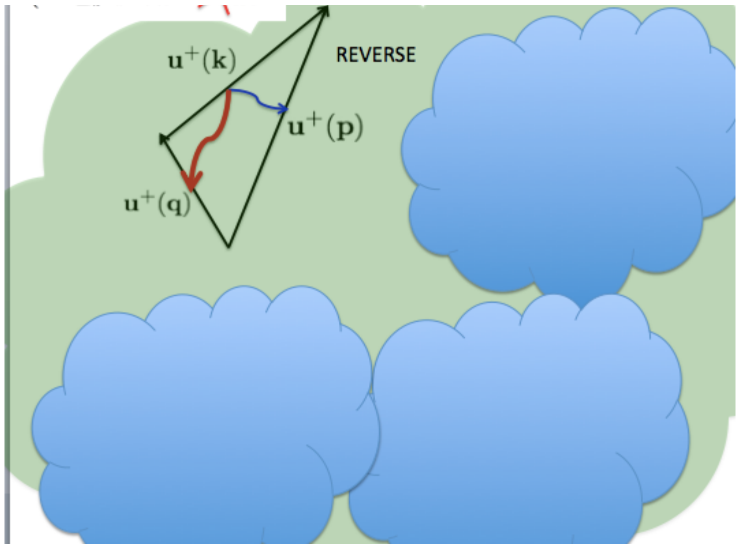
$$\mathcal{P}^{\pm} \equiv \frac{\boldsymbol{h}^{\pm} \otimes \overline{\boldsymbol{h}^{\pm}}}{\overline{\boldsymbol{h}^{\pm}} \cdot \boldsymbol{h}^{\pm}}. \qquad \boldsymbol{v}^{\pm}(\boldsymbol{x}) \equiv \sum_{\boldsymbol{k}} \mathcal{P}^{\pm} \boldsymbol{u}(\boldsymbol{k});$$

$$\boldsymbol{u}(\boldsymbol{k}) = u^{+}(\boldsymbol{k})\boldsymbol{h}^{+}(\boldsymbol{k}) + u^{-}(\boldsymbol{k})\boldsymbol{h}^{-}(\boldsymbol{k})$$

LOCAL BELTRAMIZATION (IN FOURIER)

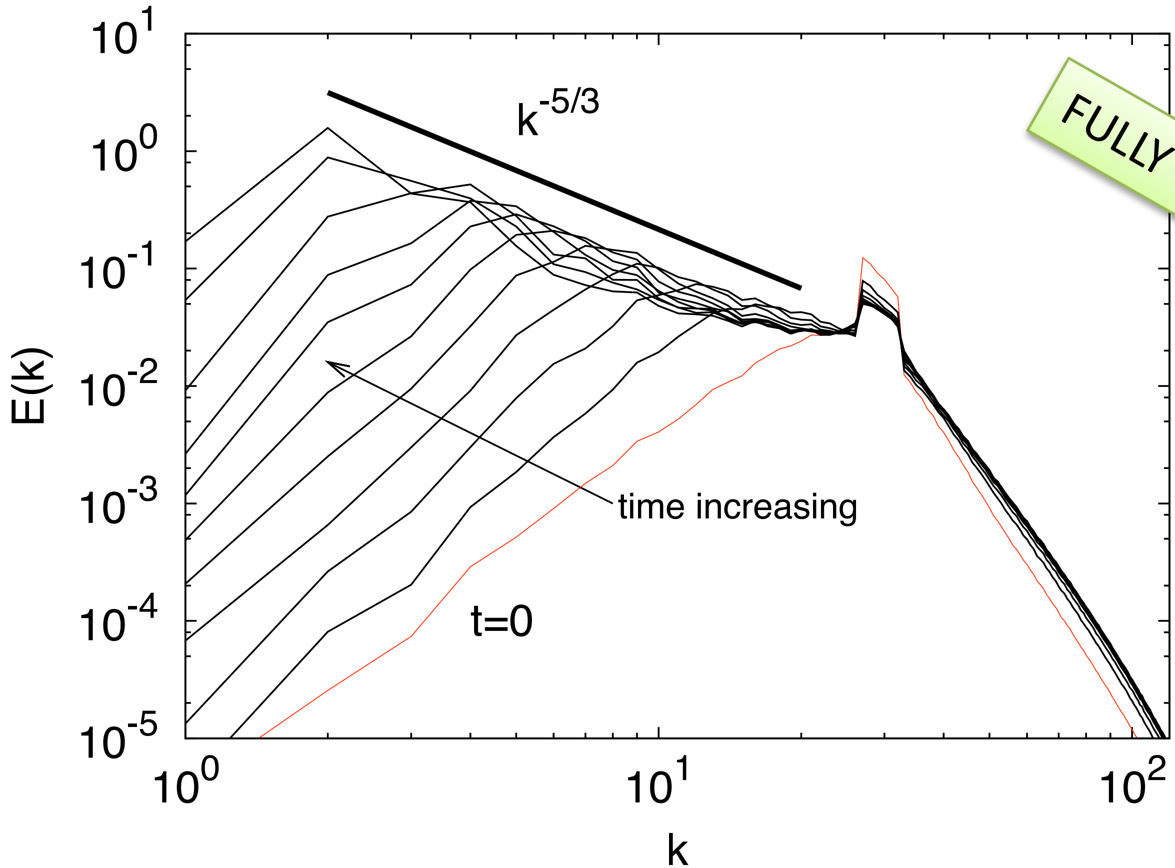
$$\partial_t v^{+} + \mathcal{P}^{+} B[v^{+}, v^{+}] = \nu \Delta v^{+} + \mathbf{f}^{+}$$

decimated-NSE

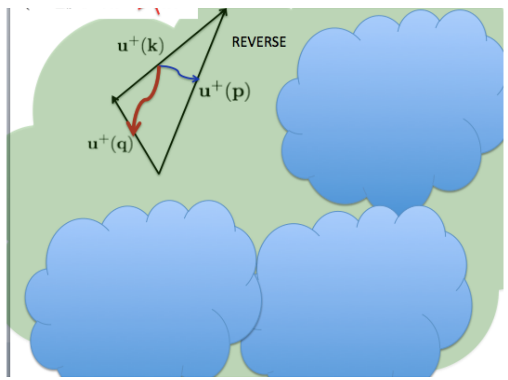


HOMOCHIRAL 3D NAVIER STOKES EQS.

$$\begin{cases} E = \sum_{\boldsymbol{k}} |u^+(\boldsymbol{k})|^2 + |u^-(\boldsymbol{k})|^2; \\ H = \sum_{\boldsymbol{k}} k(|u^+(\boldsymbol{k})|^2 - |u^-(\boldsymbol{k})|^2). \end{cases}$$



FULLY 3D AND ISOTORPIC



L.B., S. MUSACCHIO & F. TOSCHI Phys. Rev. Lett. 108 164501, 2012.

HOMOCHIRAL 3D NAVIER STOKES EQS.

ESISTENCE AND UNIQUENESS OF WEAK SOLUTIONS OF THE HELICAL-DECIMATED NSE

$$\begin{cases} \partial_t \boldsymbol{v}^+ = \mathcal{P}^+(-\boldsymbol{v}^+ \cdot \boldsymbol{\nabla} \boldsymbol{v}^+ - \boldsymbol{\nabla} p^+) + \nu \Delta \boldsymbol{v}^+ + \boldsymbol{f}^+ \\ \nabla \cdot \boldsymbol{v}^+ = 0 \end{cases}$$

HILBERT-NORM COINCIDES WITH THE SIGN-DEFINITE HELICTY

$$||g||_{H^{1/2}} = \sum_{\boldsymbol{k}} k |g(k)|^2$$

CONSERVATION HELICITY: NEW APRIORI BOUND ON THE VELOCITY

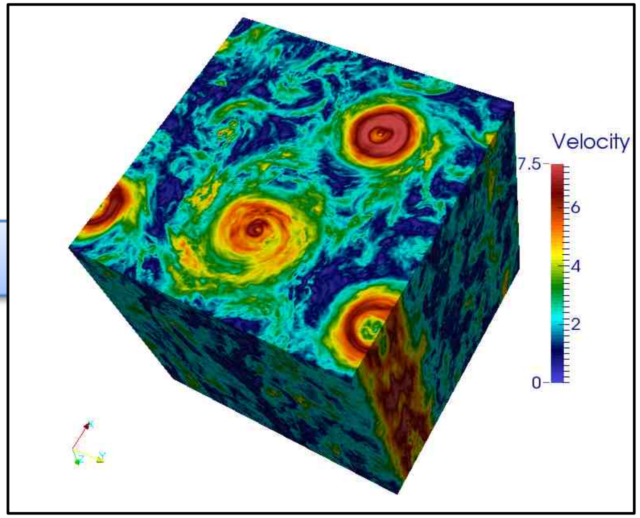
$$\frac{1}{2}\partial_t \sum_{\boldsymbol{k}} k |u^+(\boldsymbol{k},t)|^2 + \frac{\nu}{2} \sum_{\boldsymbol{k}} k^3 |u^+(\boldsymbol{k},t)|^2 \leq \frac{1}{2\nu} \sum_{\boldsymbol{k}} |f^+(\boldsymbol{k})|^2 k^{-1}.$$

$$\frac{1}{2}\partial_t ||v^+||^2_{H^{\frac{1}{2}}} + \frac{\nu}{2} ||v^+||^2_{H^{\frac{3}{2}}} \leq \frac{1}{2\nu} \sum_{\boldsymbol{k}} |f^+(\boldsymbol{k})|^2 k^{-1}.$$

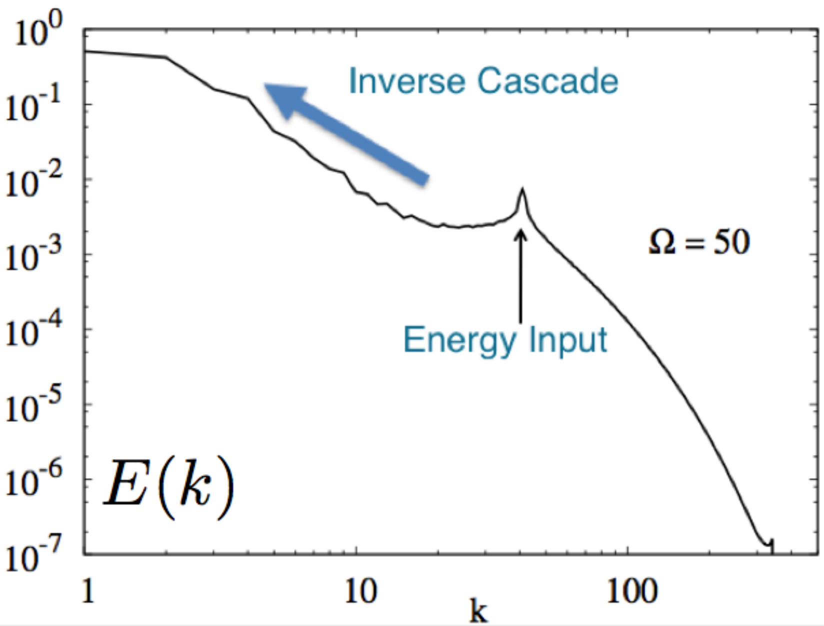
$$v^+ \in L^\infty_t H^{\frac{1}{2}}_x; \qquad \sqrt{\nu} v^+ \in L^2_t H^{\frac{3}{2}}_x$$

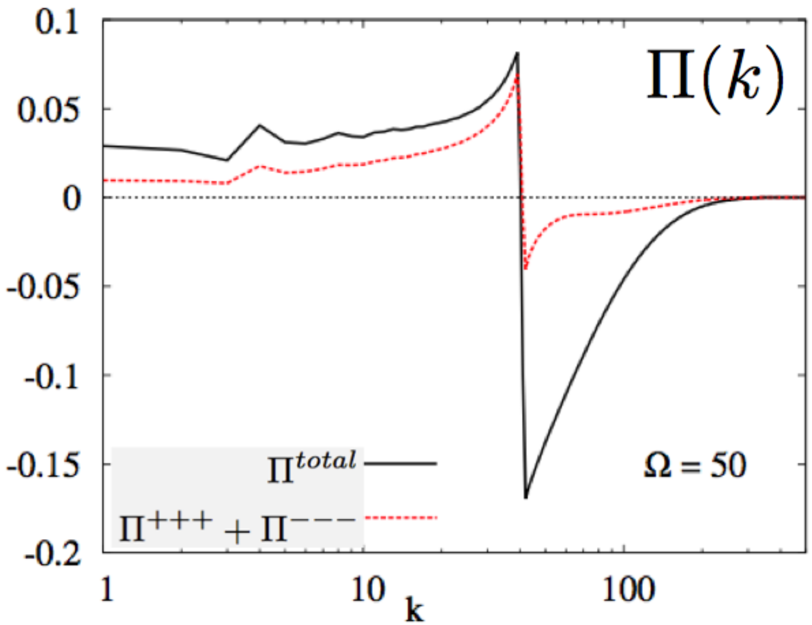
L.B. E. TITI, J. Stat. Phys. (2013)

Inverse cascade at $\Omega = 50$



Inverse flux is brought mainly by +++ and --- triads.



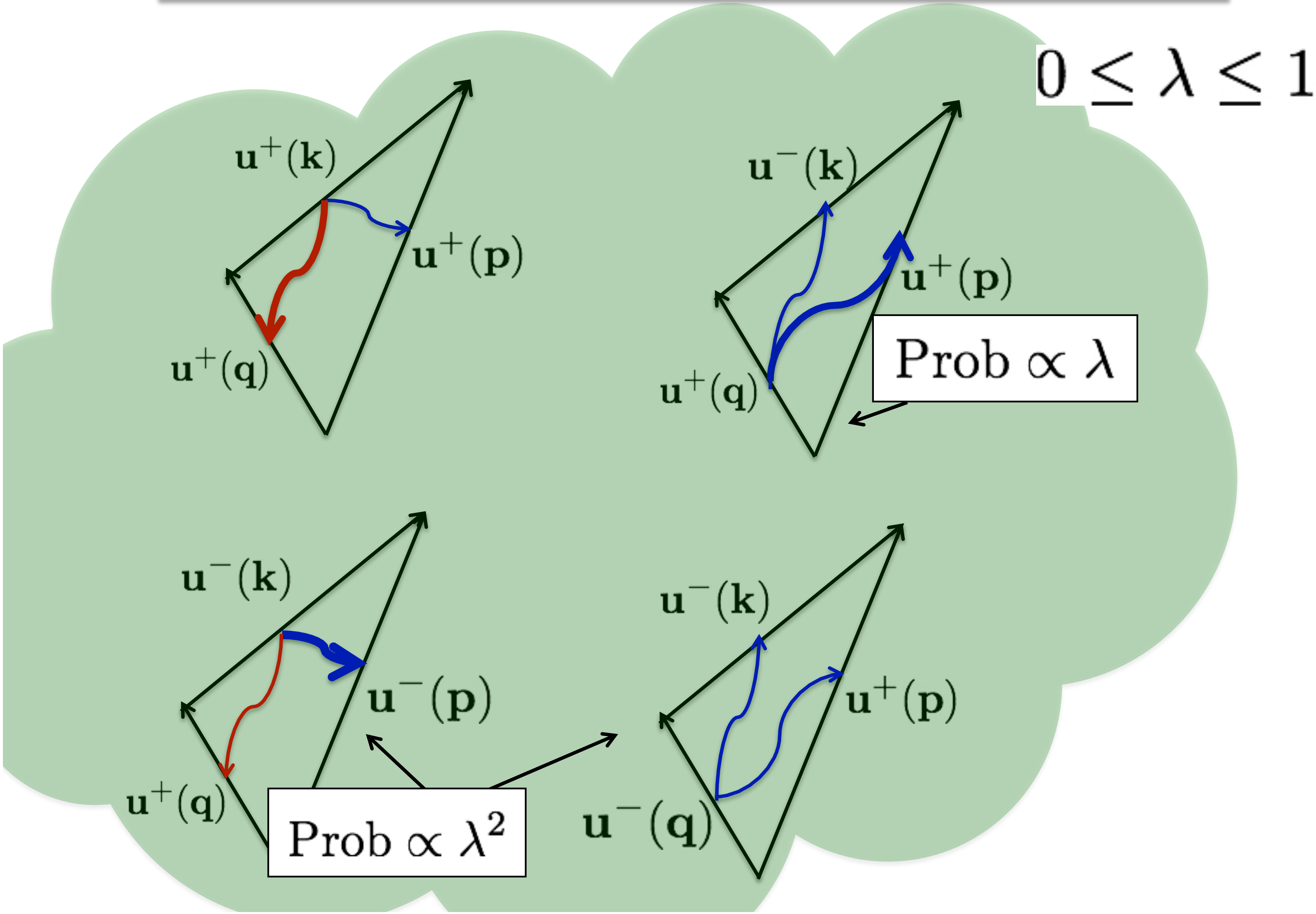


$$Ro \sim 0.15$$

$$\Pi^{(+++)}(k) = \langle u^{+}_{<k} N(u^{+}, u^{+}) \rangle$$

WITH G. SAHOO AND P. PERLEKAR (unpublished)

TRIADIC INTERACTION IN STOCHASTICALLY DECIMATED NAVIER_STOKES EQS

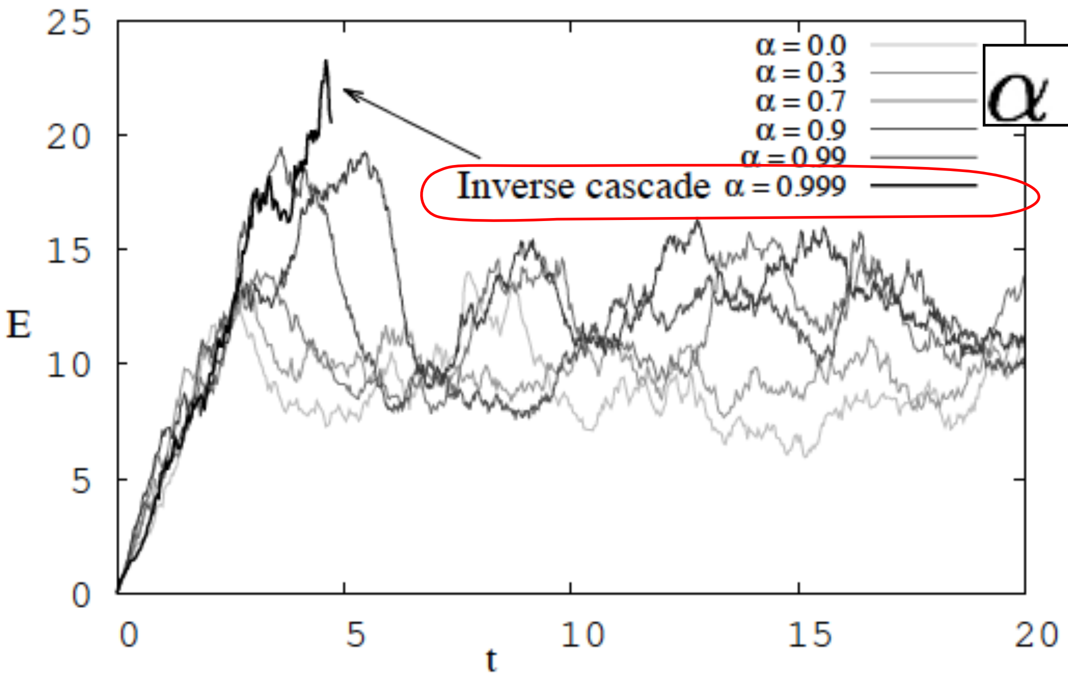


TRIADIC INTERACTION IN STOCHASTICALLY DECIMATED NAVIER_STOKES EQS

$$\boldsymbol{u}^\alpha(\boldsymbol{x}) \equiv D^\alpha \boldsymbol{u}(\boldsymbol{x}) \equiv \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}}\, \mathcal{D}^\alpha_{\boldsymbol{k}} \boldsymbol{u}_{\boldsymbol{k}}, \qquad (4)$$

where $\mathcal{D}^\alpha_{\boldsymbol{k}} \equiv (1-\gamma^\alpha_{\boldsymbol{k}})+\gamma^\alpha_{\boldsymbol{k}}\mathcal{P}^+_{\boldsymbol{k}}$ and $\gamma^\alpha_{\boldsymbol{k}} = 1$ with probability α or $\gamma^\alpha_{\boldsymbol{k}} = 0$ with probability $1-\alpha$. The α-decimated Navier-Stokes equations (α-NSE) are

$$\partial_t \boldsymbol{u}^\alpha = D^\alpha[-\boldsymbol{u}^\alpha \cdot \boldsymbol{\nabla} \boldsymbol{u}^\alpha - \boldsymbol{\nabla} p^\alpha] + \nu \Delta \boldsymbol{u}^\alpha, \qquad (5)$$



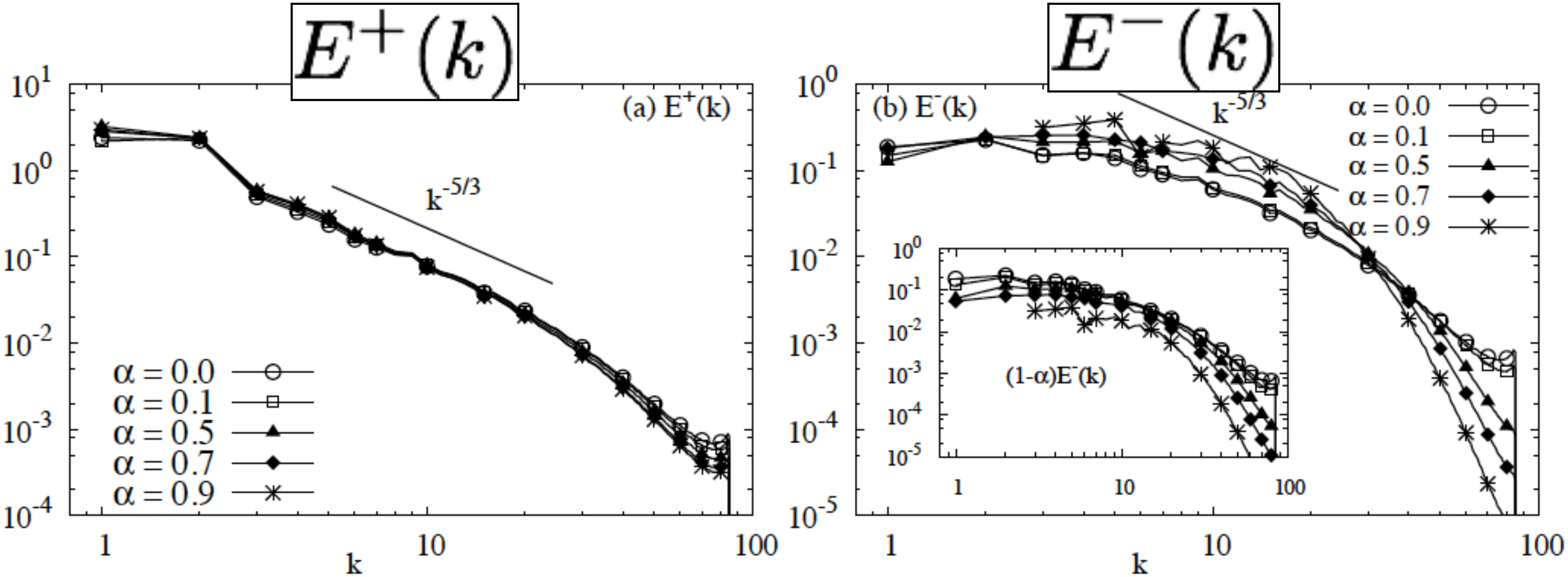
$\alpha = 1 - \lambda$

G. Sahoo, F. Bonaccorso and L. B. Phys. Rev. E (2015)

TRIADIC INTERACTION IN STOCHASTICALLY DECIMATED NAVIER_STOKES EQS

$$E(k) = E^{+}(k) + E^{-}(k)$$

$$H(k) = k(E^{+}(k) - E^{-}(k))$$

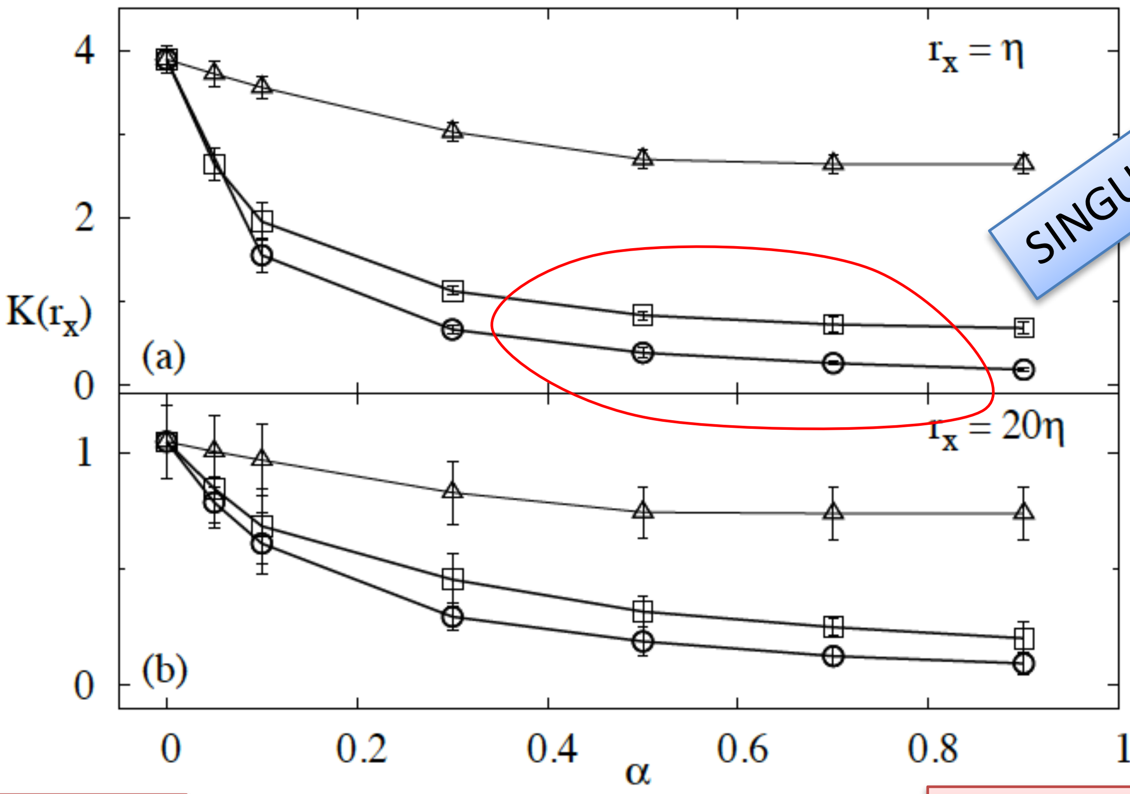


RECOVERY OF MIRROR SYMMETRY

G. Sahoo, F. Bonaccorso and L. B.
Phys. Rev. E (2015)

TRIADIC INTERACTION IN STOCHASTICALLY DECIMATED NAVIER_STOKES EQS

$$K(r) = \frac{\langle(\delta_r v)^4\rangle}{\langle(\delta_r v)^2\rangle^2} - 3$$

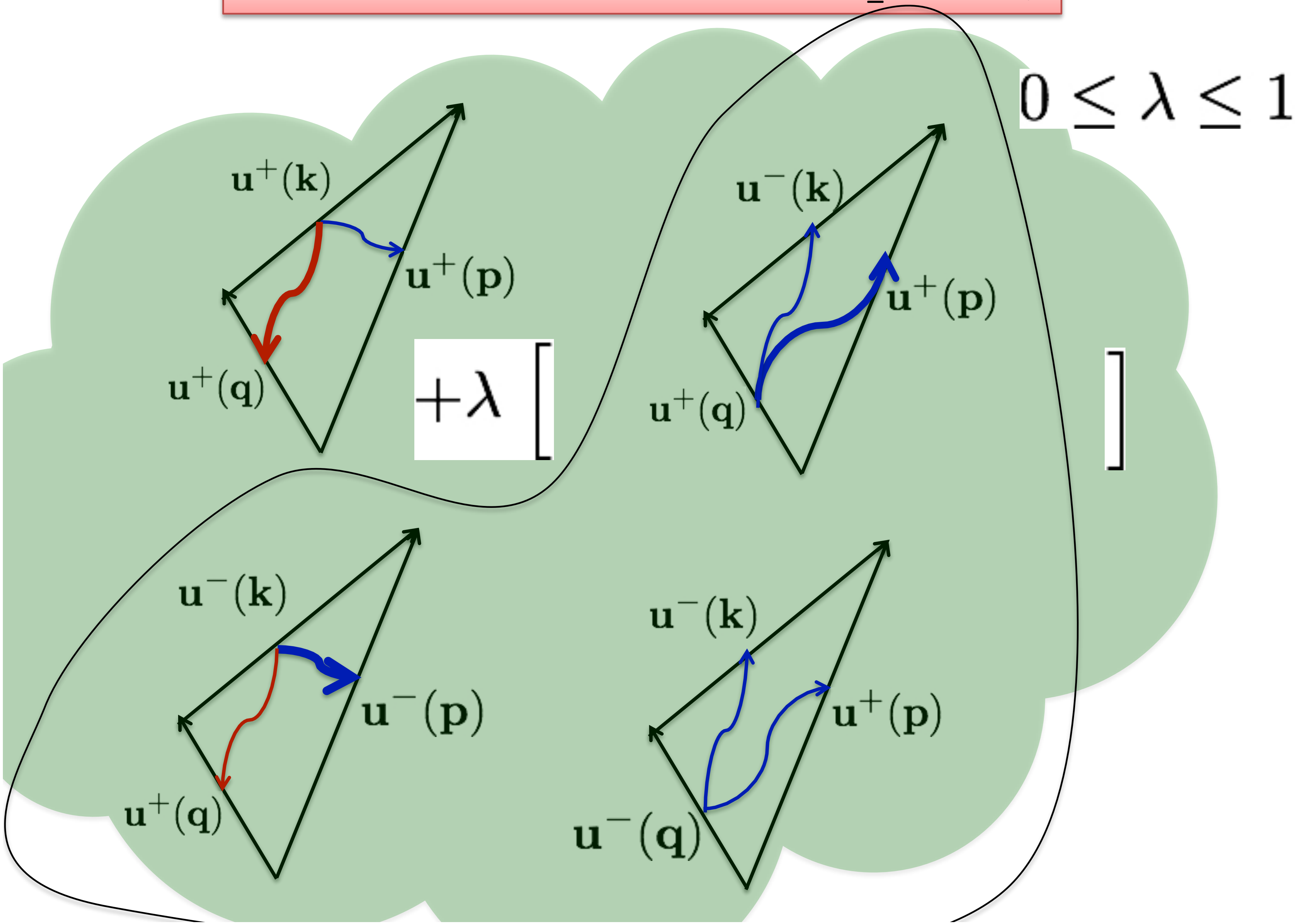


SINGULAR EFFECT ON INTERMITTENCY

G. Sahoo, F. Bonaccorso and L. B.
Phys. Rev. E (2015)

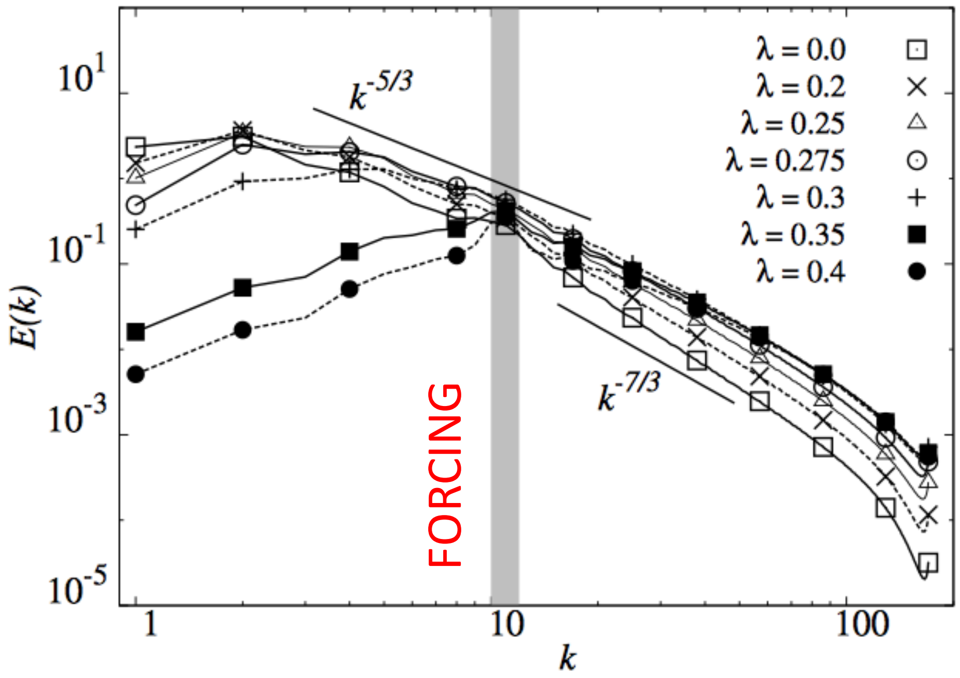
FULL NS ⟷ HOMO CHIRAL

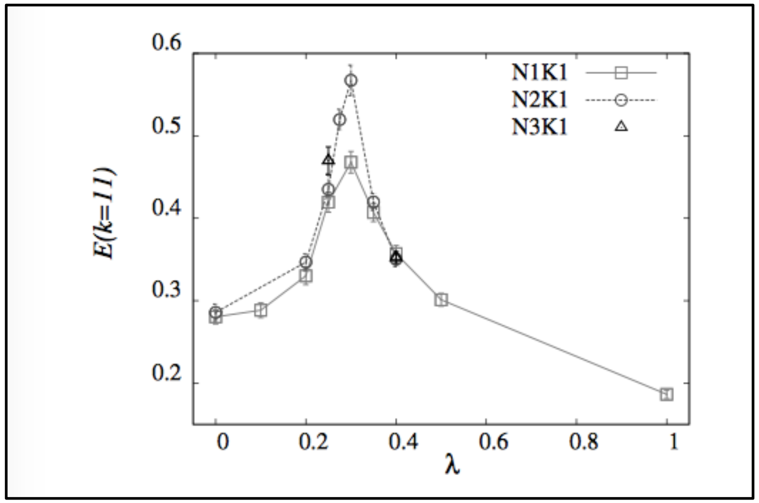
TRIADIC INTERACTION IN REWEIGTHED NAVIER_STOKES EQS



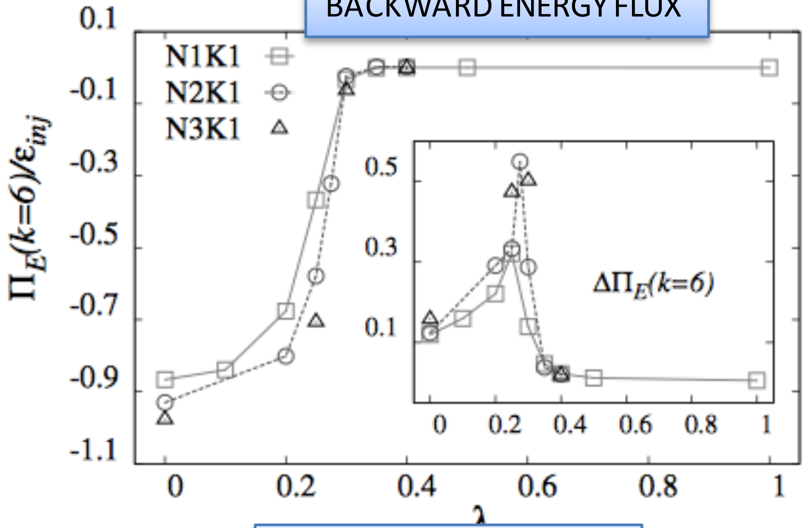
TRIADIC INTERACTION IN REWEIGHTED NAVIER_STOKES EQS

$$\mathcal{N} = \lambda(\mathbf{u}\times\mathbf{w})+(1-\lambda)[\mathbb{P}^{+}(\mathbf{u}^{+}\times\mathbf{w}^{+})+\mathbb{P}^{-}(\mathbf{u}^{-}\times\mathbf{w}^{-})]$$

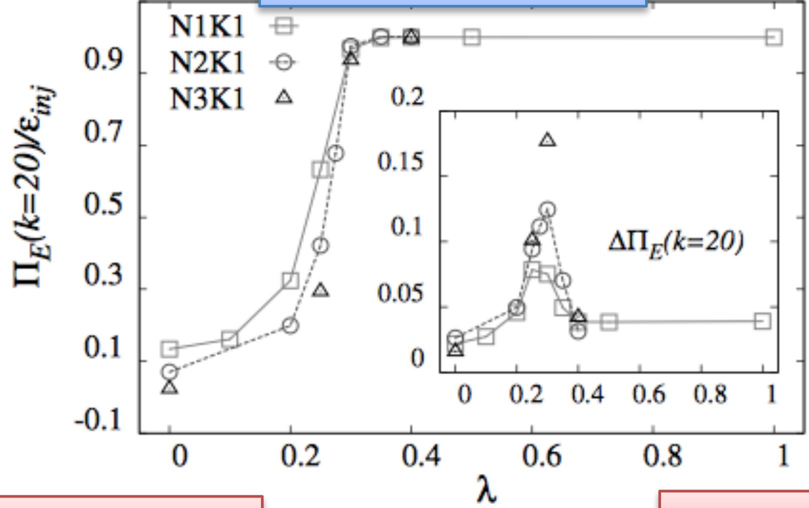




BACKWARD ENERGY FLUX



FORWARD ENERGY FLUX



WITH G. SAHOO AND A. ALEXAKIS (PRL **118**, 164501, 2017)

HOMO CHIRAL ⟷ FULL NS

CONCLUSIONS

ROLE OF HELICITY IN THE FORWARD/BACKWARD 3D ENERGY TRANSFER (FOURIER)

ROLE OF HOMO-CHIRAL TRIADS VISIBLE ALSO IN ROTATING TURBULENCE

EXISTENCE OF A SHARP PHASE-TRANSITION BAKWARD/FORWARD IF SOME NON-LINEAR INTERACTION ARE REWEIGHTED

HETERO-CHIRAL TRIADS PLAY A SINGULAR ROLE FOR INTERMITTENCY IF PARTICIPATING WITH THE CORRECT PREFACTOR

IMPLICATION FOR REGULARITY OF SOLUTIONS

IMPLICATION FOR SMALL AND LARGE SCALE DYNAMO

IMPLICATION FOR REAL-SPACE INTERMITTENCY AND ENERGY BACKSCATTER?

PNAS

Helicity and singular structures in fluid dynamics

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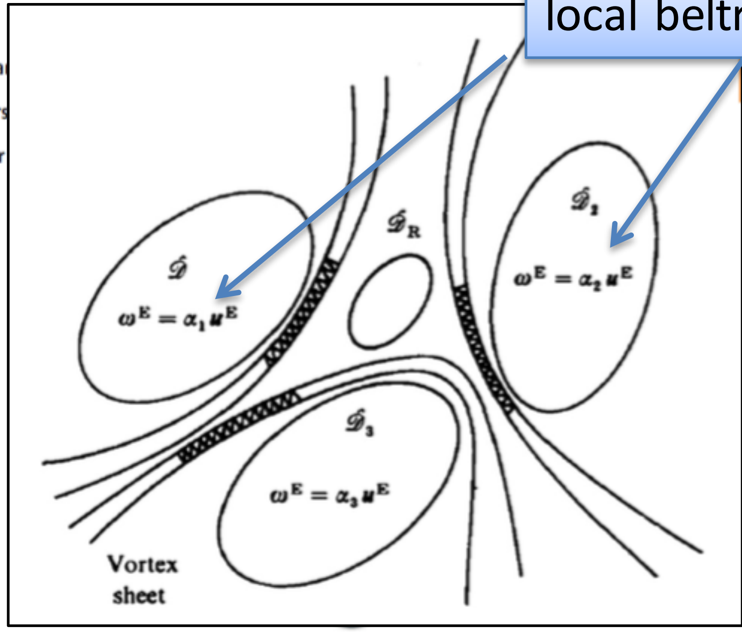
Department of Applied Mathematics and Theoretical Physics, University of Ca

This contribution is part of the special series of Inaugural Articles by members

Contributed by H. Keith Moffatt, January 14, 2014 (sent for review December

Helicity is, like energy, a quadratic invariant of the Euler equations of ideal fluid flow, although, unlike energy, it is not sign definite. In physical terms, it represents the degree of linkage of the vortex lines of a flow, conserved when conditions are such that these vortex lines are frozen in the fluid. Some basic properties of helicity are reviewed, with particular reference to (*i*) its crucial role in the dynamo excitation of magnetic fields in cosmic systems; (*ii*) its bearing on the existence of Euler flows of arbitrarily complex streamline topology; (*iii*) the constraining role of the analogous magnetic helicity in the determination of stable knotted minimum-energy magnetostatic structures; and (*iv*) its role in depleting nonlinearity in the Navier-Stokes equations, with implications for the coherent structures and energy cascade of turbulence. In a final section, some singular phenomena in low Revnolds number flows are brieflv described.



PNAS

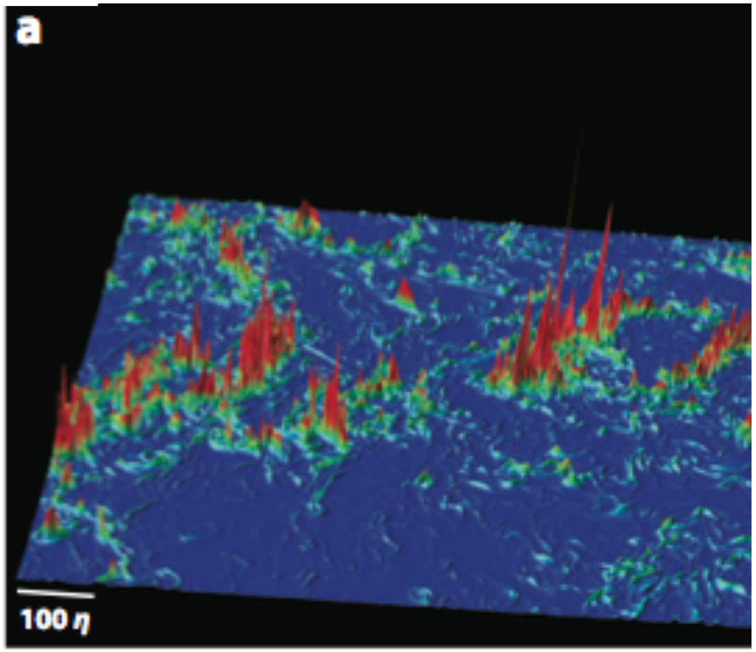
Helicity conservation by flow across scales in reconnecting vortex links and knots

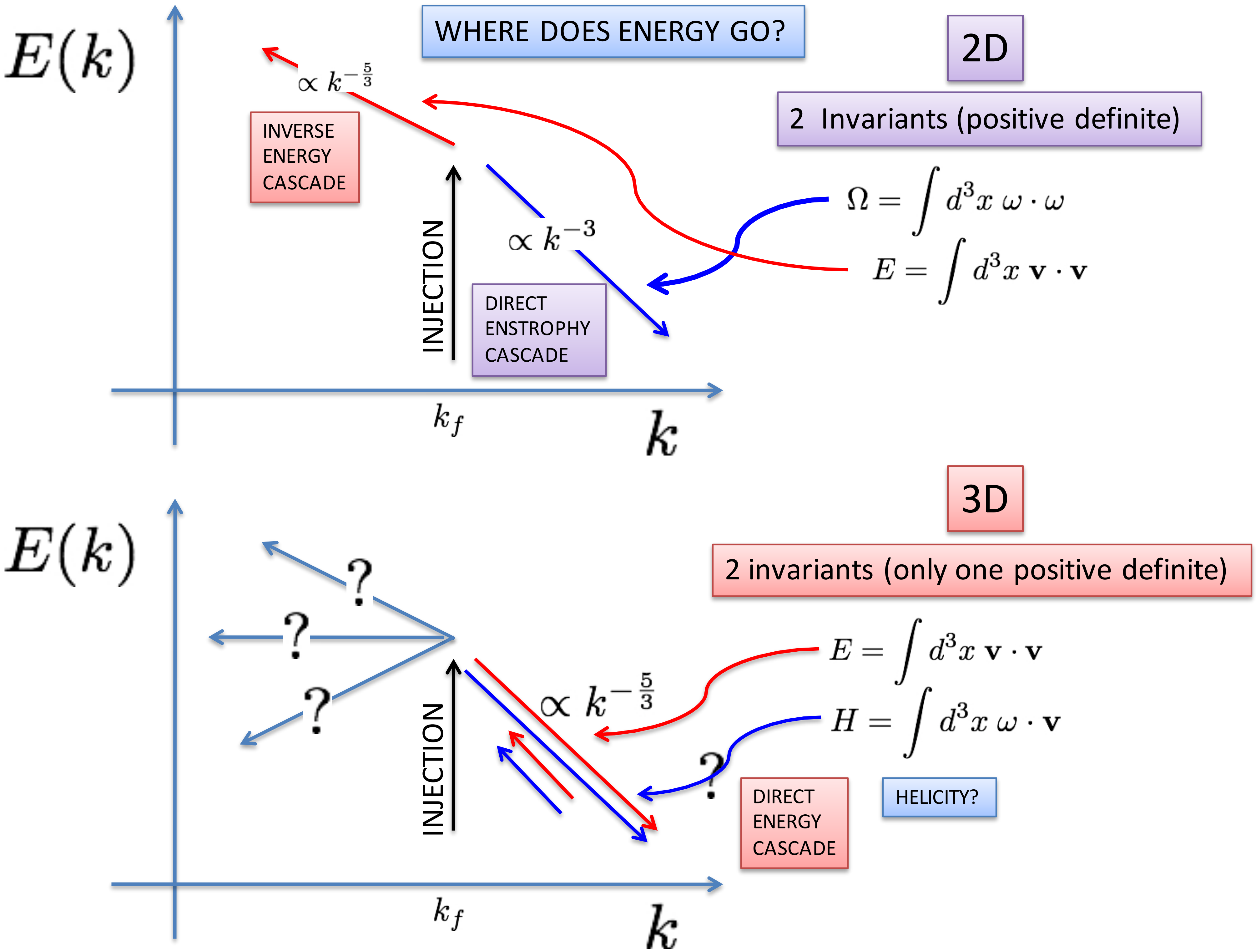
Martin W. Scheeler[a,1,2], Dustin Kleckner[a,1,2], Davide Proment[b], Gordon L. Kindlmann[c], and William T. M.

[a]James Franck Institute, Department of Physics, and [c]Computation Institute, Department of Computer Science, The University of Chicag and [b]School of Mathematics, University of East Anglia, Norwich Research Park, Norwich NR4 7TJ, United Kingdom

Edited* by Leo P. Kadanoff, The University of Chicago, Chicago, IL, and approved August 28, 2014 (received for review April 19, 2014)

The conjecture that helicity (or knottedness) is a fundamental conserved quantity has a rich history in fluid mechanics, but the nature of this conservation in the presence of dissipation has proven difficult to resolve. Making use of recent advances, we create vortex knots and links in viscous fluids and simulated superfluids and track their geometry through topology-changing reconnections. We find that the reassociation of vortex lines through a reconnection enables the transfer of helicity from links and knots to helical coils. This process is remarkably efficient, owing to the antiparallel orientation spontaneously adopted by the reconnecting vortices. Using a new method for quantifying the spatial helicity spectrum, we find that the reconnection process can be viewed as transferring helicity between scales, rather than dissipating it. We also infer the presence of geometric deformations that convert helical coils into even smaller scale twist, where it may ultimately be dissipated. Our results suggest that helicity conservation plays an important role in fluids and related fields, even in the presence of dissipation.




WHERE DOES ENERGY GO?
2D
2 Invariants (positive definite)
$\Omega = \int d^3x\, \omega \cdot \omega$
$E = \int d^3x\, \mathbf{v} \cdot \mathbf{v}$
$E(k)$
$\propto k^{-\frac{5}{3}}$
INVERSE ENERGY CASCADE
INJECTION
$\propto k^{-3}$
DIRECT ENSTROPHY CASCADE
k_f
k
3D
2 invariants (only one positive definite)
$E = \int d^3x\, \mathbf{v} \cdot \mathbf{v}$
$H = \int d^3x\, \omega \cdot \mathbf{v}$
$E(k)$
?
?
?
INJECTION
$\propto k^{-\frac{5}{3}}$
?
DIRECT ENERGY CASCADE
HELICITY?
k_f
k

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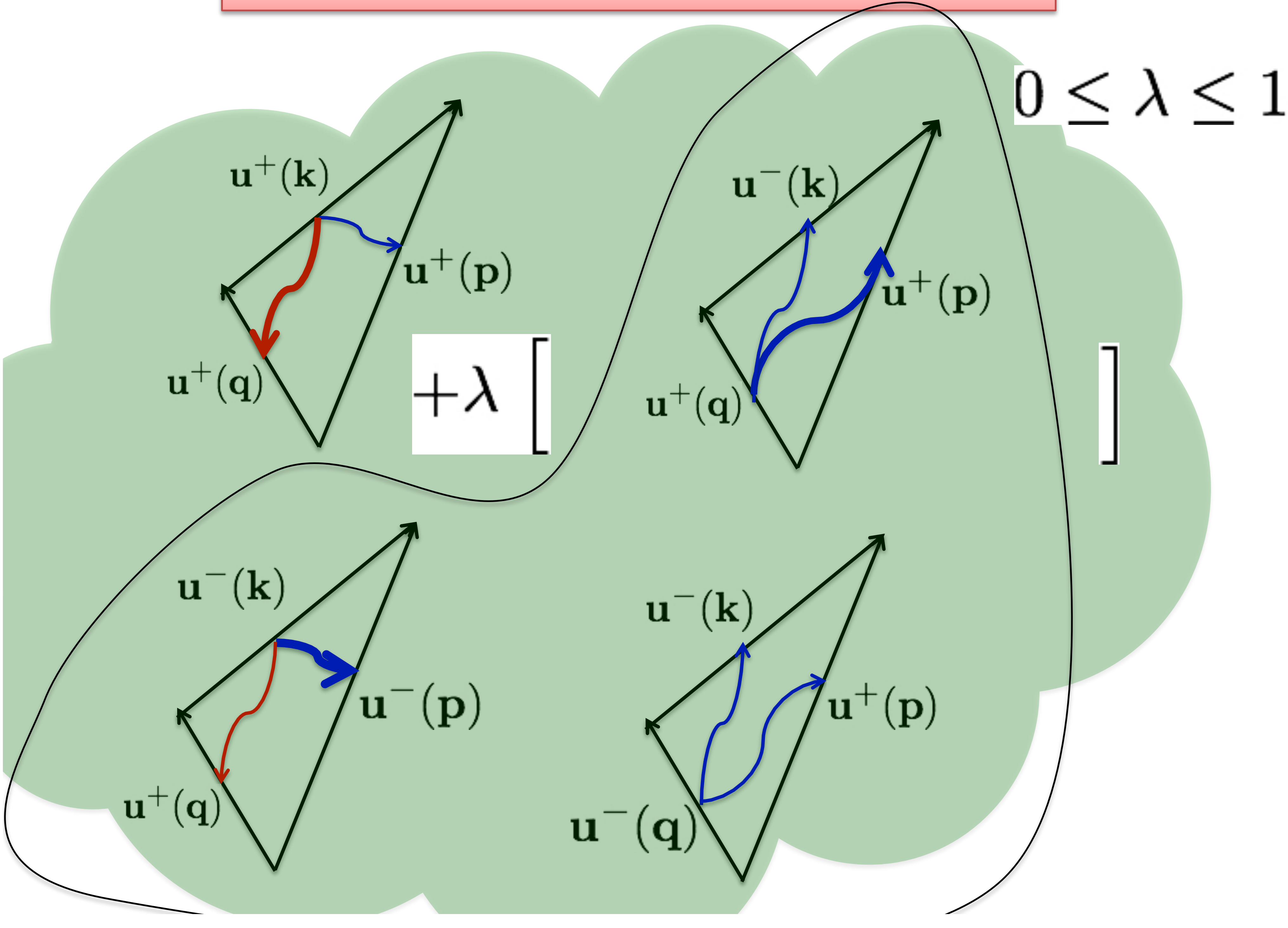
On the Global Regularity of a Helical-Decimated Version of the 3D Navier-Stokes Equations

Authors | Authors and affiliations

Luca Biferale, Edriss S. Titi

We study the global regularity, for all time and all initial data in $H^{1/2}$, of a recently introduced decimated version of the incompressible 3D Navier-Stokes (dNS) equations. The model is based on a projection of the dynamical evolution of Navier-Stokes (NS) equations into the subspace where helicity (the L^2-scalar product of velocity and vorticity) is sign-definite. The presence of a second (beside energy) sign-definite inviscid conserved quadratic quantity, which is equivalent to the $H^{1/2}$-Sobolev norm, allows us to demonstrate global existence and uniqueness, of space-periodic solutions, together with continuity with respect to the initial conditions, for this decimated 3D model. This is achieved thanks to the establishment of two new estimates, for this 3D model, which show that the $H^{1/2}$ and the time average of the square of the $H^{3/2}$ norms of the velocity field remain finite. Such two additional bounds are known, in the spirit of the work of H. Fujita and T. Kato (Arch. Ration. Mech. Anal. 16:269–315, 1964; Rend. Semin. Mat. Univ. Padova 32:243–260, 1962), to be sufficient for showing well-posedness for the 3D NS equations. Furthermore, they are directly linked to the helicity evolution for the dNS model, and therefore with a clear physical meaning and consequences.

TRIADIC INTERACTION IN REWEIGTHED NAVIER_STOKES EQS



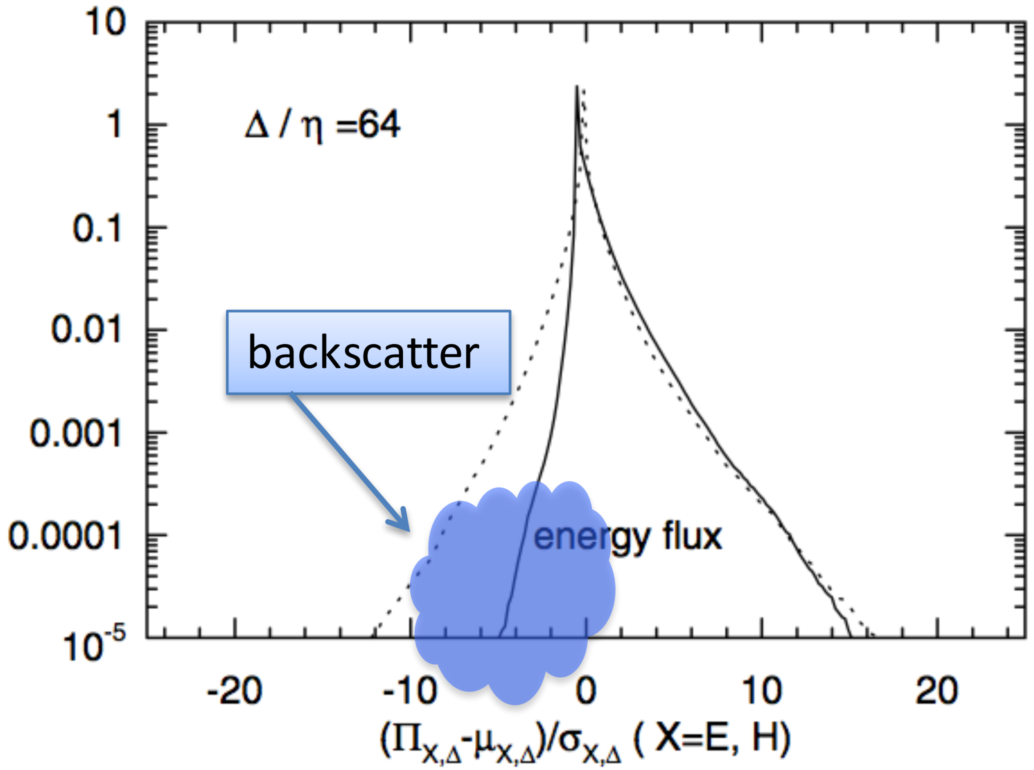
$$\partial_t \overline{\boldsymbol{v}} + (\overline{\boldsymbol{v}} \cdot \nabla)\overline{\boldsymbol{v}} = -\nabla \overline{p} - \nabla \cdot \tau(\boldsymbol{v}, \boldsymbol{v}) + \nu \Delta \overline{\boldsymbol{v}}$$

SUB GRID /REYNOLDS STRESS: $\tau_{ij}(\boldsymbol{v}, \boldsymbol{v}) = \overline{v_i v_j} - \overline{v}_i \overline{v}_j$

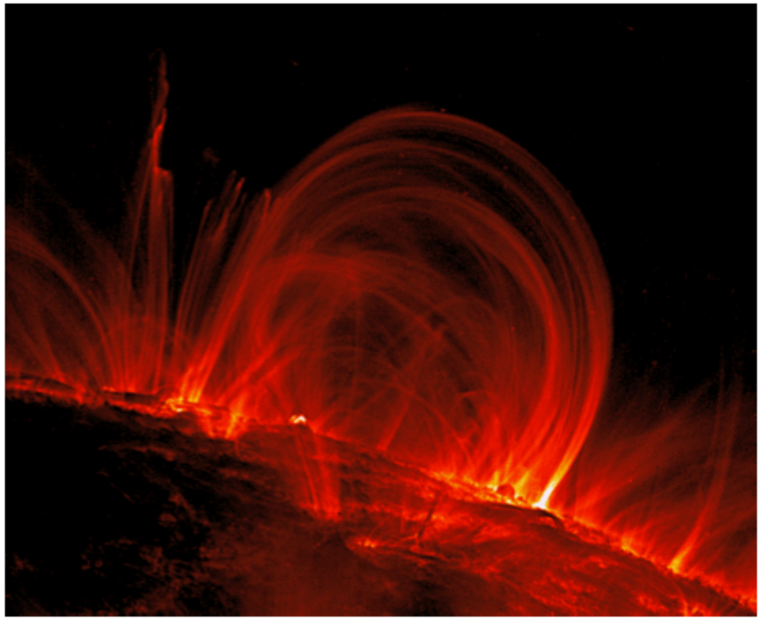
$$\partial_t \frac{1}{2} \overline{v}_i \overline{v}_i + \partial_j A_j = -\Pi$$

SUB GRID ENERGY TRANSFER:

$$\Pi = -\tau_{ij} \overline{\boldsymbol{S}}_{ij} \qquad \overline{\boldsymbol{S}}_{ij} = \frac{1}{2}(\partial_i \overline{v}_j + \partial_j \overline{v}_i)$$



Large-scale magnetic fields in MHD



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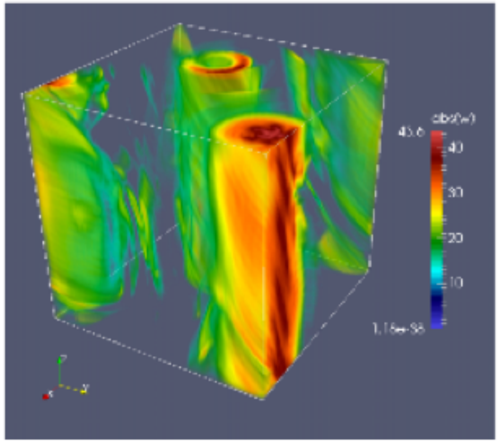
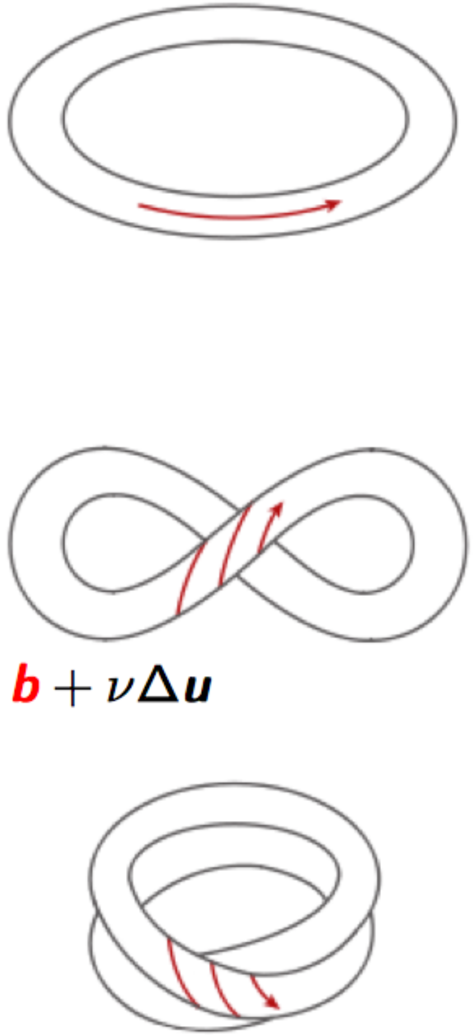


Dallas & Alexakis PoF 2015



$$\partial_t \boldsymbol{u} = -\frac{1}{\rho}\nabla P - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}(\nabla \times \boldsymbol{b}) \times \boldsymbol{b} + \nu\Delta\boldsymbol{u}$$

$$\partial_t \boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} + \eta\Delta\boldsymbol{b}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{and} \quad \nabla\cdot\boldsymbol{b} = 0$$

$$H_m(t) = \int_V d\boldsymbol{x}\ \boldsymbol{a}(\boldsymbol{x},t)\cdot\boldsymbol{b}(\boldsymbol{x},t) \rightarrow \text{ inverse cascade}$$

$$H_k(t) = \int_V d\boldsymbol{x}\ \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\omega}(\boldsymbol{x},t) \rightarrow \text{ dynamo action (e.g. } \alpha\text{-effect)}$$

WITH M. LINKMANN, G. SAHOO, M MCKAY AND A. BERERA AJP 2017

Helical Fourier decomposition

$$(\partial_t + \nu k^2) u_{\boldsymbol{k}}^{s_k *} = \frac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \Bigg(\sum_{s_p, s_q} g^{s_k}_{s_p s_q} (s_p p - s_q q) u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q}$$

$$- \sum_{\sigma_p, \sigma_q} g^{s_k}_{\sigma_p \sigma_q} (\sigma_p p - \sigma_q q) b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q} \Bigg) ,$$

LORENTZ

$$(\partial_t + \eta k^2) b_{\boldsymbol{k}}^{\sigma_k *} = \frac{\sigma_k k}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \Bigg(\sum_{\sigma_p, s_q} g^{\sigma_k}_{\sigma_p s_q} b_{\boldsymbol{p}}^{\sigma_p} u_{\boldsymbol{q}}^{s_q} - \sum_{s_p, \sigma_q} g^{\sigma_k}_{s_p \sigma_q} u_{\boldsymbol{p}}^{s_p} b_{\boldsymbol{q}}^{\sigma_q} \Bigg)$$

ADVECTION + STRETCHING

$$E_{kin} = \sum_k |u_k^+|^2 + |u_k^-|^2 \qquad H_{kin} = \sum_k k(|u_k^+|^2 - |u_k^-|^2)$$

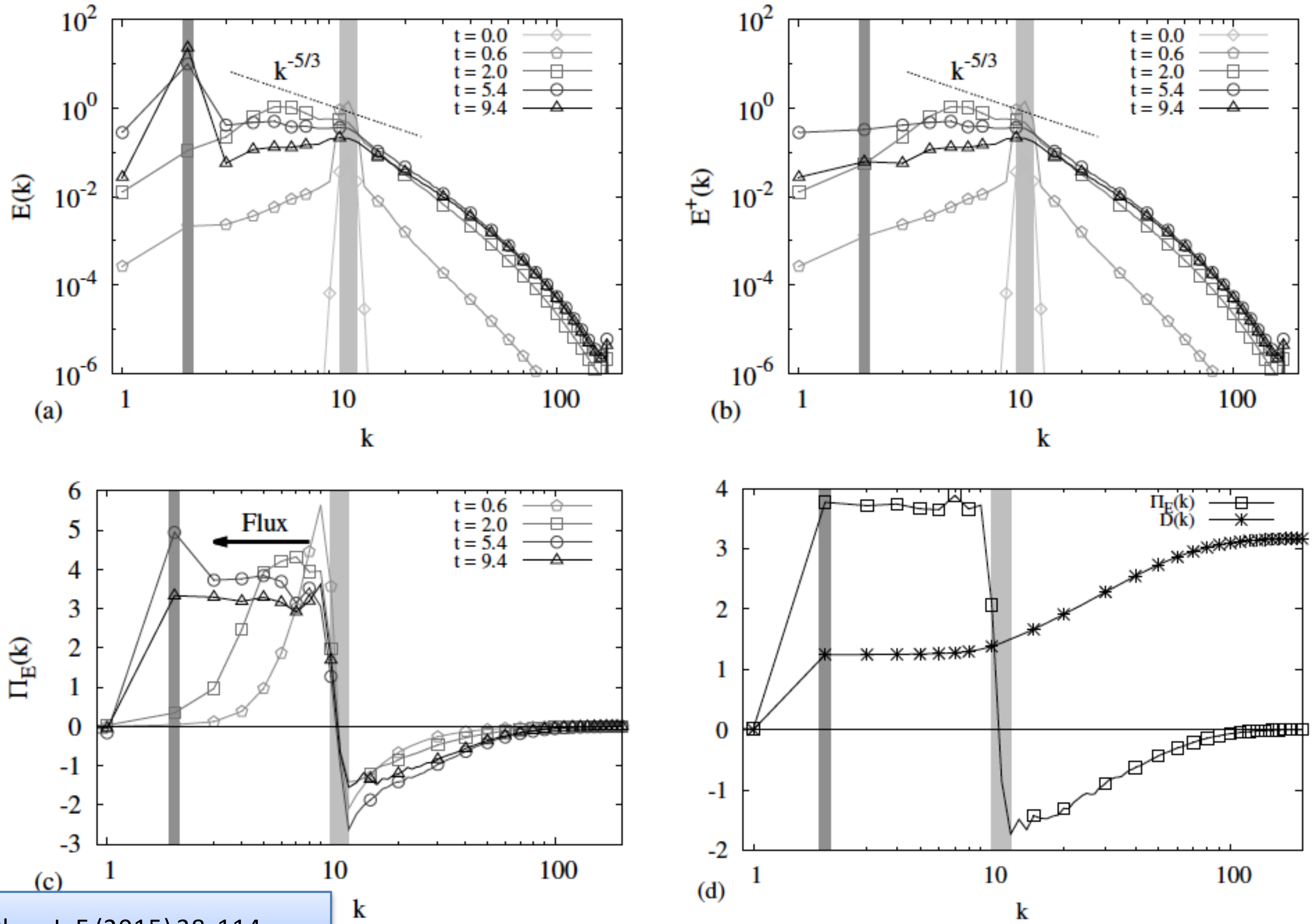
$$E_{mag} = \sum_k (|b_k^+|^2 + |b_k^-|^2) \qquad H_{mag} = \sum_k k^{-1}(|b_k^+|^2 - |b_k^-|^2)$$

Stability analysis

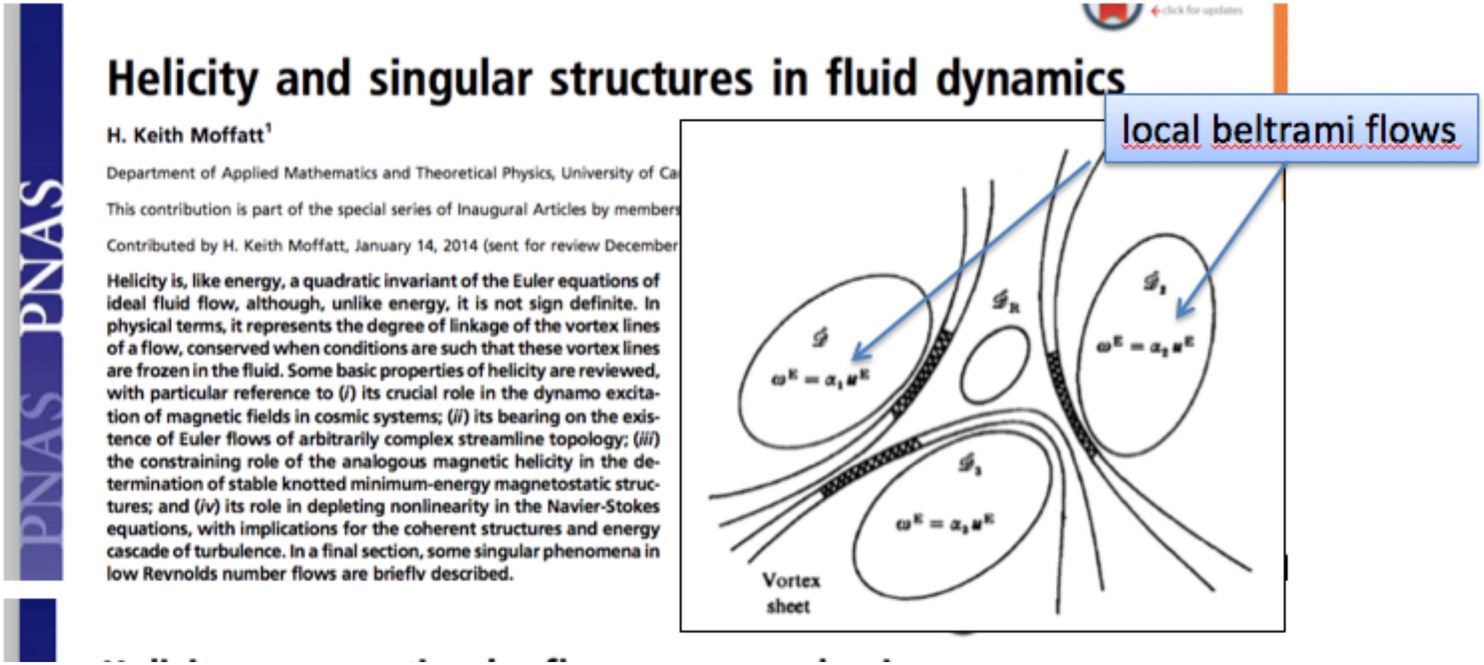
$$
\begin{aligned}
\partial_t u_{\boldsymbol{k}}^{s_k *} &= g^{s_k}_{s_p s_q}(s_p p - s_q q)\, u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} - g^{s_k}_{\sigma_p \sigma_q}(\sigma_p p - \sigma_q q)\, b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q} \\
\partial_t u_{\boldsymbol{p}}^{s_p *} &= g^{s_p}_{s_q s_k}(s_q q - s_k k)\, u_{\boldsymbol{q}}^{s_q} u_{\boldsymbol{k}}^{s_k} - g^{s_p}_{\sigma_q \sigma_k}(\sigma_q q - \sigma_k k)\, b_{\boldsymbol{q}}^{\sigma_q} b_{\boldsymbol{k}}^{\sigma_k} \\
\partial_t u_{\boldsymbol{q}}^{s_q *} &= g^{s_q}_{s_k s_q}(s_k k - s_p p)\, u_{\boldsymbol{k}}^{s_k} u_{\boldsymbol{q}}^{s_q} - g^{s_q}_{\sigma_k \sigma_p}(\sigma_k k - \sigma_p p)\, b_{\boldsymbol{k}}^{\sigma_k} b_{\boldsymbol{p}}^{\sigma_p} \\
\partial_t b_{\boldsymbol{k}}^{\sigma_k *} &= \sigma_k k\, \left(g^{\sigma_k}_{\sigma_p s_q} b_{\boldsymbol{p}}^{\sigma_p} u_{\boldsymbol{q}}^{s_q} - g^{\sigma_k}_{s_p \sigma_q} u_{\boldsymbol{p}}^{s_p} b_{\boldsymbol{q}}^{\sigma_q} \right) \\
\partial_t b_{\boldsymbol{p}}^{\sigma_p *} &= \sigma_p p\, \left(g^{\sigma_p}_{\sigma_q s_k} b_{\boldsymbol{q}}^{\sigma_q} u_{\boldsymbol{k}}^{s_k} - g^{\sigma_p}_{s_q \sigma_k} u_{\boldsymbol{q}}^{s_q} b_{\boldsymbol{k}}^{\sigma_k} \right) \\
\partial_t b_{\boldsymbol{q}}^{\sigma_q *} &= \sigma_q q\, \left(g^{\sigma_q}_{\sigma_k s_p} b_{\boldsymbol{k}}^{\sigma_k} u_{\boldsymbol{p}}^{s_p} - g^{\sigma_q}_{s_k \sigma_p} u_{\boldsymbol{k}}^{s_k} b_{\boldsymbol{p}}^{\sigma_p} \right)
\end{aligned}
$$

TRIAD-BY-TRAID BACKWARD -> HELICAL CONDENSATE ON THE MINORITY MODES

$$u'(x) \equiv \mathcal{D}_m u(x) \equiv \sum_{k} e^{ikx} \left[(1-\gamma_k) + \gamma_k \mathcal{P}_k^{+}\right] \hat{u}_k,$$



G. Sahoo, L.B. Eur. Phys. J. E (2015) 38: 114.



Helicity and singular structures in fluid dynamics

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This contribution is part of the special series of Inaugural Articles by members

Contributed by H. Keith Moffatt, January 14, 2014 (sent for review December

Helicity is, like energy, a quadratic invariant of the Euler equations of ideal fluid flow, although, unlike energy, it is not sign definite. In physical terms, it represents the degree of linkage of the vortex lines of a flow, conserved when conditions are such that these vortex lines are frozen in the fluid. Some basic properties of helicity are reviewed, with particular reference to (*i*) its crucial role in the dynamo excitation of magnetic fields in cosmic systems; (*ii*) its bearing on the existence of Euler flows of arbitrarily complex streamline topology; (*iii*) the constraining role of the analogous magnetic helicity in the determination of stable knotted minimum-energy magnetostatic structures; and (*iv*) its role in depleting nonlinearity in the Navier-Stokes equations, with implications for the coherent structures and energy cascade of turbulence. In a final section, some singular phenomena in low Reynolds number flows are briefly described.