Anomalous Transport in Metapopulation Networks

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Content

1. INTRODUCTION
   - Scale-Free Networks
   - Metapopulation Transport on Networks
2. FRACTIONAL TRANSPORT EQUATIONS ON NETWORKS
   - Axiom of Cumulative Inertia
   - Fractional Diffusion Equation and Subdiffusion
   - Anomalous Aggregation on Network
   - U-form Residence Time PDF and Empirical Evidence

Transport in Metapopulation Networks

Scale-Free Network: The power-law probability that a given node has k links (order k) to other nodes: $P(k) \sim k^{-\gamma}, \ \gamma \in [2, 3]$.

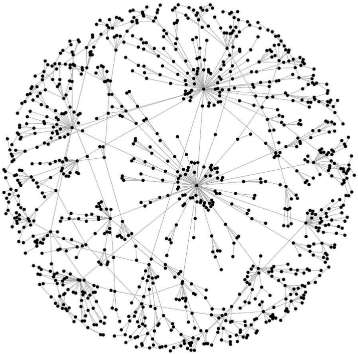

Figure : Barabási-Albert network,

$^{(*)}$ Colizza and Vespignani, Phys. Rev. Lett. **99**, 148701 (2007).

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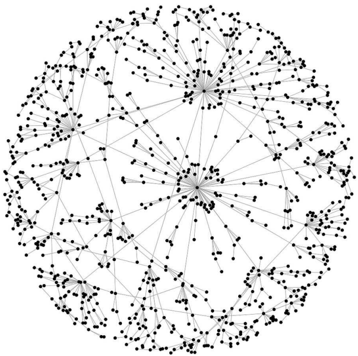

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Mean field transport equation:

$$\frac{dN_k(t)}{dt} = -\mathbb{I}_k(t) + k\sum_{k'} P(k'|k)\frac{\mathbb{I}_{k'}(t)}{k'}$$

N_k : mean number of individuals in node of order k;

$\mathbb{I}_k$: mean flux out of node of order k;

$P(k'|k)$: the probability of a link between nodes of order $k \to k'$. For $\mathbb{I}_k(t) = \lambda N_k(t)$:

$$N_k^{st} = k\frac{\langle N \rangle}{\langle k \rangle}$$

– well-connected nodes are more populous.(*)

Human activity is not Poissonian!(†)

(†)A.-L. Barabási,

Axiom of Cumulative Inertia in Network Theory

Axiom of Cumulative Inertia:

An individual's escape probability from a node decreases with the (residence) time T spent in the node.

This is an empirical sociological law. The escape rate γ_k decreases with residence time

$$\gamma_k(\tau) = \frac{\mu_k}{\tau + \tau_0}, \quad \mu_k, \tau_0 > 0$$

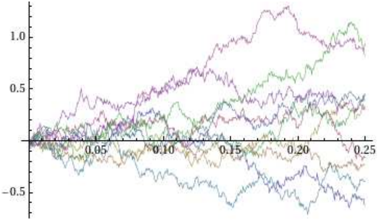
Probability density function (PDF) of a residence time is

$$\psi_k(\tau) = \frac{\mu_k}{\tau + \tau_0} \left(\frac{\tau_0}{\tau + \tau_0} \right)^{\mu_k} \sim 1/\tau^{1+\mu_k},$$

Fedotov and Stage, Phys. Rev. Lett. **118**, 9 (2017).

Anomalous subdiffusive transport:

Mean square displacement of Brownian particle: $< B^2(t) >= 2Dt$



Macroscopic transport equation:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2}, \qquad x \in \mathbb{R}$$

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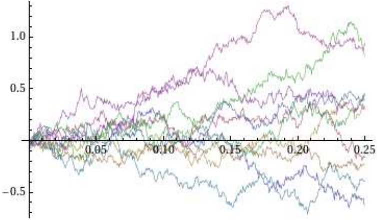

Macroscopic transport equation:

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}, \qquad x \in \mathbb{R}$$

Mean square displacement for subdiffusion:

$$< X^2(t) > \sim t^{\mu} \qquad 0 < \mu < 1$$

What is the macroscopic equation for the concentration ρ?

Anomalous transport: fractional order PDE

Macroscopic equation for the concentration ρ:

$$\frac{\partial \rho}{\partial t} = D_\mu \frac{\partial^2}{\partial x^2}\left(\mathcal{D}_t^{1-\mu}\rho\right), \tag{1}$$

where the Riemann-Liouville (fractional) derivative $\mathcal{D}_t^{1-\mu}$ is defined as

$$\mathcal{D}_t^{1-\mu}\rho = \frac{1}{\Gamma(\mu)}\frac{\partial}{\partial t}\int_0^t \frac{\rho\left(x,u\right)du}{(t-u)^{1-\mu}} \tag{2}$$

Anomalous transport: fractional order PDE

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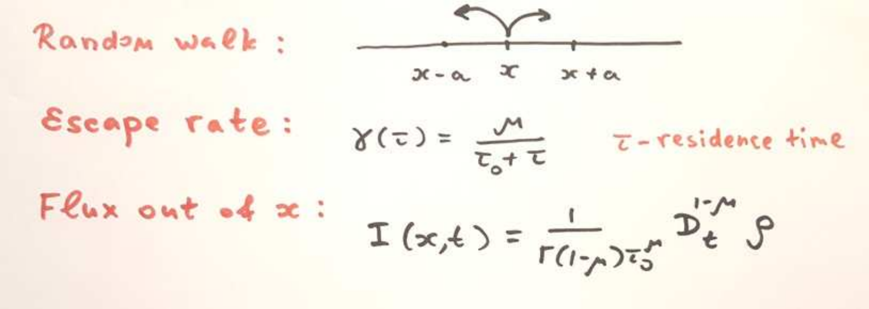
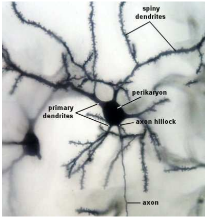
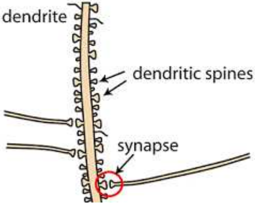


Anomalous subdiffusion: $< X^2(t) > \sim t^{\mu} \qquad 0 < \mu < 1$

- Subdiffusion is due to trapping inside dendritic spines





Non-Markovian behavior of particles performing random walk occurs when particles are trapped during the random time with non-exponential distribution.

Power law waiting time distribution

$$\phi(t) \sim \frac{1}{t^{1+\mu}}$$

with $0 < \mu < 1$ as $t \to \infty$.

The mean waiting time is infinite.

How Does the Axiom of Cumulative Inertia Affect the Flux?

Instead of the classical flux $\mathbb{I}_k = \lambda N_k(t)$, the Axiom of Cumulative Inertia leads to a flux

$$\mathbb{I}_k = \frac{1}{\Gamma(1-\mu_k)\tau_0^{\mu_k}} \mathcal{D}_t^{1-\mu_k} N_k(t), \quad \text{for} \quad \mu_k < 1,$$

$\mathcal{D}_t^{1-\mu_k}$ is the Riemann-Liouville fractional derivative.
Classical results are transient with no steady state.

Main result: ultimately all individuals are attracted to the node with $\mu_k < 1$.

The node's mean residence time $\langle T \rangle \to \infty$ (anomalous trapping).

Anomalous Aggregation

What is happening inside the trapping (anomalous) node?

Consider the structural density of individuals at time t with residence time τ, $n_{trap}(t, \tau)$.

$$n_{trap}(t,\tau) \to \frac{N}{\Gamma(1-\mu_k)\Gamma(\mu_k)\tau^{\mu_k}(t-\tau)^{1-\mu_k}},$$

N is the total number of individuals in the network.

$\rightarrow$ Most individuals have been there for a long time, or are new arrivals.

Is this realistic?

Yes! Data: American MidWest

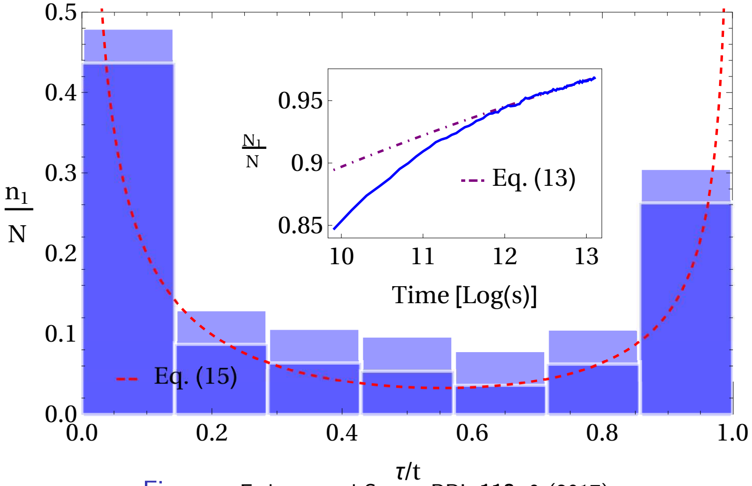


Figure : Fedotov and Stage, PRL **118**, 9 (2017)

Conclusions

• The mesoscopic description of non-Markovian reaction-transport processes on the network is still an open problem.