Modeling Quantum Boundary Energy by Power-Law Potentials

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Principal recent collaborators

Advanced undergraduate students:

- Steven Murray and Colin Whisler
- Joseph Merritt and Thomas Settlemyre

University of Oklahoma and its diaspora:

- Kim Milton
- Prachi Parashar
- Pushpa Kalauni

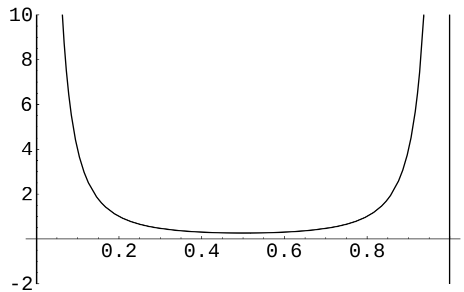
Motivations

Roots

- Semiclassical gravity: The expectation value of the stress-energy-momentum tensor should act as the source of the gravitational field.
- Casimir effect: Long-distance van der Waals forces can be calculated from the energy in the electromagnetic field as a function of geometrical parameters. **Can this energy be localized?**

The scalar field near a flat reflecting plate

- Total energy shows a divergence absent in the EM case.
- Similar divergences arise in EM for curved plates.
- Energy density calculations show the offending energy is concentrated at the surface of the plate.



The cutoff approach

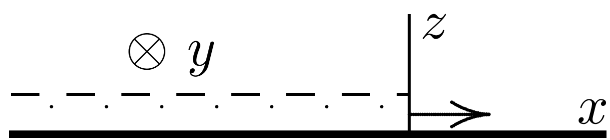
Presumably the Dirichlet boundary condition (or perfect conduction, in EM) is unrealistic. Real conductors are not perfect at high frequencies.

So, one might insert a factor $e^{-\omega\tau}$ in the integral over frequencies. This gives finite and initially plausible results. However:

There is a **pressure anomaly**!

Principle of virtual work: The force on a test surface should equal the decrease in the total energy behind the surface as it moves a unit distance:

$$F = -dE/dL, \quad \text{or} \quad p \equiv \langle T_{xx} \rangle = -u \equiv -\langle T_{tt} \rangle.$$



Without the cutoff, that is formally true, although the integrals for F and E diverge.

But with the cutoff, one gets

$$F = +\frac{1}{2}\frac{dE}{dL} \quad \left(\text{not } (-1)\frac{dE}{dL}\right).$$

One may think of τ as it. That is, the cutoff involves a Wick rotation of the Green function in the time coordinate. Effectively this introduces a spurious time dependence of $G \equiv \langle \phi(\underline{x})\phi(\underline{x}')\rangle$ so that the time derivatives in E come out wrong.

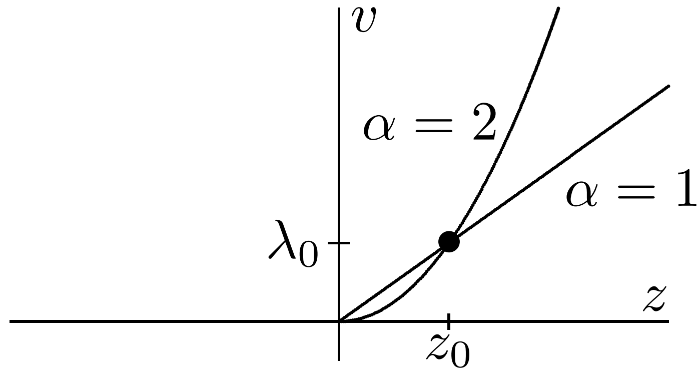
A soft wall

Surely the anomaly would not arise in a model that is nonsingular from the start, with consistent Lagrangian equations of motion.

For calculational ease, we study the scalar field with an external scalar potential (or z-dependent mass):

$$\Box\varphi = V\varphi, \qquad V(\mathbf{r}) = \begin{cases} 0, & z < 0, \\ z^\alpha, & z > 0. \end{cases}$$

This gives an increasingly steep wall near $z = 1$ as $\alpha \to \infty$.



In properly dimensioned variables, z^α becomes $\lambda_0(z/z_0)^\alpha$ (α = positive integer).

There is one independent length, $(z_0^\alpha/\lambda_0)^{1/(\alpha+2)}$.

(Bouas, . . . , Wagner, arXiv:1106.1162)

The components of the [VEV of the] stress tensor come from the **reduced Green function** and its second derivatives.

$$\left(-\frac{\partial^2}{\partial z^2} + V(z) + \kappa^2\right) g_\kappa(z, z') = \delta(z - z').$$

And g can be constructed from basis solutions

$$\left(-\frac{\partial^2}{\partial z^2} + V(z) + \kappa^2\right) \left\{ {F \atop G} \right\} = 0,$$

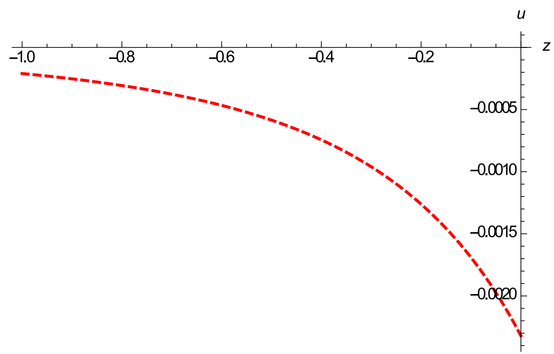
with $F(0) = 1$, $\lim_{z\to+\infty} F(z) = 0$, and some arbitrary independent choice for G.

(Milton, Phys. Rev. D **84** (2011) 065028)

Outside the wall ($z < 0$)

Here the calculations were rather easy.

Energy density for $\alpha = 6$:



(Murray, Whisler, et al., Phys. Rev. D **93** (2016) 105010)

Inside the wall ($z > 0$): Renormalization

In addition to various technical difficulties, local divergences proportional to V, V^2, $\Box V$ must be isolated and renormalized. These can be handled within the WKB approximation.

The renormalization theory in an arbitrary smooth scalar potential can be developed by dimensional regularization from the Schwinger–DeWitt expansion (Mazzitelli, Nery, & Satz, 2011). But ...

For methodological consistency

- We use point-splitting in the spirit of Christensen (1976) and Wald (1978) in curved space.
- We apply one-dimensional WKB theory to a potential that is an (arbitrary) function of z alone.

One gets divergent terms depending on δ^{-4}, $\delta^{-2}V(z)$, $V^2 \ln(V\delta^2)$, $V'' \ln(V\delta^2)$, and some finite direction-dependent terms of the same order in V and V''.
($\delta \to 0$ covariantly generalizes the τ of before.)

All terms proportional to V^2 and V'' are potentially ambiguous. They must be adjusted covariantly to preserve the conservation law, $\partial_\mu T^{\mu\nu} = -\frac{1}{2}\phi^2\partial^\nu V$. As in gravity, this forces a **trace anomaly**,

$$\langle T^\mu_\mu \rangle + V\langle\phi^2\rangle - 3\left(\xi - \tfrac{1}{6}\right)\partial_z{}^2\langle\phi^2\rangle = \frac{1}{16\pi^2}\,a_2\,,$$

where the heat kernel coefficient $a_2 = \frac{1}{2}\left(V^2 - \frac{1}{3}V''\right)$.

All direction-independent, covariant terms in $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ (proportional to 1, V, V^2, $\Box V$) can be absorbed into "bare" terms in an equation of motion for V as a dynamical field and the corresponding Einstein equation, respectively. This is so regardless of whether V is taken to be a Klein-Gordon field (Fulling et al. 2012) or the square of such a field (Mazzitelli et al. 2011).

However, the renormalized energy density contains terms like $V^2 \ln(V/\mu^2)$ that do not approach 0 inside the wall (where $z \to \infty$, $V(z) \to \infty$). They can be eliminated near any particular z by choosing $\mu = V(z)$. This is reminiscent of the **renormalization group** in perturbative QFT, but occurring in position space instead of momentum space.

(Milton, Fulling, et al., Phys. Rev. D **93** (2016) 085017)

Inside the wall ($z > 0$): Evaluation

Careful subtraction of the leading WKB terms under the integral sign produces convergent integrals for the finite remainder in the energy density and the pressure. Both outside and inside, as predicted, there is no indication of a pressure anomaly.

However, to carry out such computations all the way to numerical results requires non-WKB methods to approximate the integrands in the regimes of small κ and small z. That's where we are now.

(Merritt, Settlemyre, Fulling, Lujan, in progress)

We do our computations in *Mathematica*. (This simple model should not require advanced computational resources.) Possible methods:

- Numerical solution of the ordinary differential equation for the basis functions (**slow** (for the machine); conceptually uninformative).
- Analytical approximations to the basis functions and normalization constants (usually less accurate; slow (for the humans) — a year of work, still unfinished).

We are working to adapt the perturbative and spline methods that worked well in the exterior region.

- WKB at large κ
- Perturbation (power series) at small κ
- Some kind of smooth spline in between.

(Since we must integrate over κ to get $u(z)$ and $p(z)$, it is natural to work at fixed z and vary κ.)

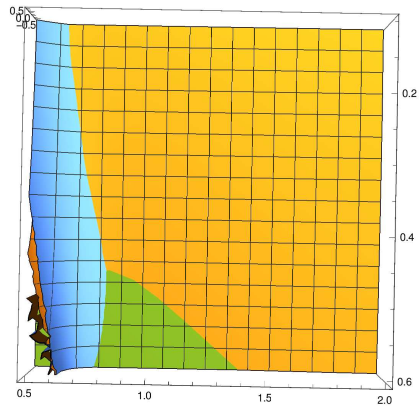
Well, it turned out they didn't always work as well as they did.

In addition to the basis functions $F(\kappa, z)$ and $G(\kappa, z)$ we need two auxiliary functions of κ alone:

- A normalization factor $c_F(\kappa)$ that relates the prototypical WKB behavior $w^{-1/4}e^{-\int w^{1/2}}$ at infinity to the chosen normalization of F at the origin (where WKB has broken down).
- A scattering coefficient $\gamma_-(\kappa)$ that relates the solution H that decays as $z \to -\infty$ (needed in the Green function construction) to a solution G that has nice initial data at $z = 0$.

For all these quantities we try to join the perturbative solution at small κ to the WKB solution at large κ by a bridging function $A\kappa + B$
(or $f(A\kappa + B)$ with some function f of more plausible concavity)
on the interval $d_1 < \kappa < d_2$, with A, B, d_1, d_2 chosen to make the function and its derivative continuous at both endpoints.
Solve linear equations for A and B, then find d_1 and d_2 by Newton's method applied to two nonlinear equations.

Search for a "triple point" where the two nonlinear functions are zero simultaneously. (Green = 0 plane)



Looks like we have a zero near $(d_1, d_2) = (.46, .82)$. Apply Newton's method to get an improved point.

Sometimes this works well.
(blue = pert, orange = WKB, green = linear spline)

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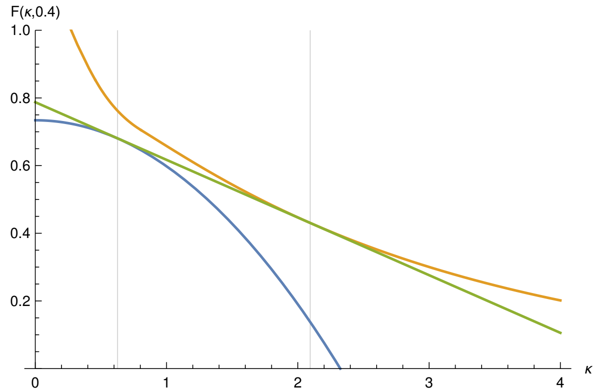
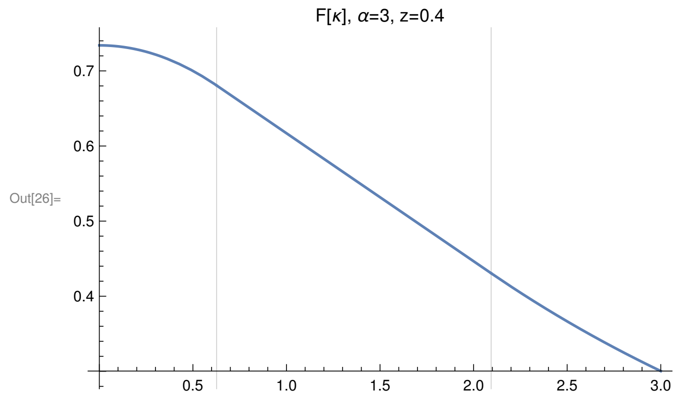
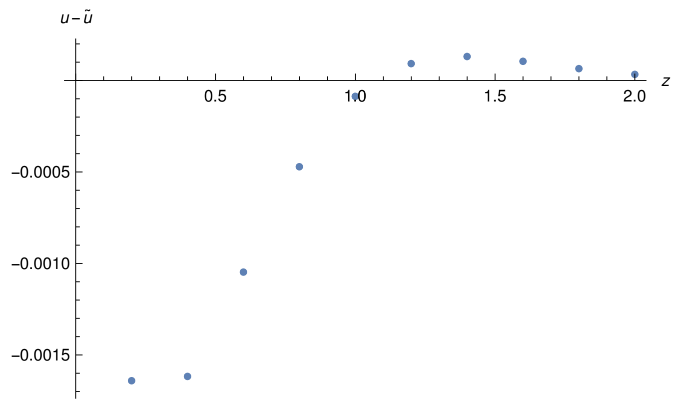


So we adopt this smooth, piecewise defined function:



But sometimes the spline procedure works horribly. (So we try another $f(A\kappa + B)$, or look for a different approach.)

To have a result to show, we reverted to the purely numerical method (which has some problems of its own). Tommy Settlemyre says, "This is something that resembles the correct renormalized energy density."



Thanks to the organizers of this conference, and to all of you for listening patiently to this progress report. This is a toy problem, but we are determined to solve it completely and well. Maybe one more year . . .