R-matrix, star-triangle relations and Yangians for conformal algebras

A. P. Isaev [1]

[1]**Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Russia**

Based on joint works with D.Chicherin, S.Derkachov, J. Fuksa, D.Karakhanyan and R.Kirschner
arXiv:1206.4150, 1303.4929, 1511.06152, 1612.04713

Ginzburg Centennial Conference on Physics
May 29 - June 3, Lebedev Institute, Moscow, Russia

Plan

1. **Introduction.**
   - Star-triangle relations (STR) and multiloop calculations
   - Yang-Baxter $\mathcal{R}$-operators
2. **$\mathcal{R}$-operators and L-operators**
   - $\mathcal{R}$- and L-operators for conformal algebra
   - Notion of Yangians and their applications
   - Spinor and differential reps of $conf(\mathbb{R}^{p,q})$ and $Y(conf(\mathbb{R}^{p,q}))$
3. **Explicit form of spinorial R-matrix and L-operator**

Star-triangle relation (STR)

To evaluate multi-loop Feynman integrals we have to consider integral

$$\int \frac{d^D z}{(x-z)^{2\alpha}\, z^{2\beta}\, (z-y)^{2\gamma}} \; .$$

where $x, y, z \in \mathbb{R}^D$, $x = (x_1, \dots, x_D), \dots, x^{2\beta} = (x_\mu x^\mu)^\beta$.
Interesting special case $\alpha + \beta + \gamma = D$ firstly considered in CFT (see e.g. E.S.Fradkin, M.Ya.Palchik, Phys. Rep. 1978)

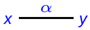
$$\int \frac{d^D z}{(x-z)^{2\alpha'}\, z^{2(\alpha+\beta)}\, (z-y)^{2\beta'}} = \frac{G(\alpha,\beta)}{(x)^{2\beta}\, (x-y)^{2(\frac{D}{2}-\alpha-\beta)}\, (y)^{2\alpha}} \; ,$$

where parameters $\boxed{\alpha' := \frac{D}{2} - \alpha}$, $\Rightarrow \alpha' + (\alpha+\beta) + \beta' = D$,

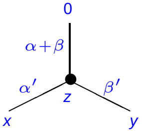
$$G(\alpha,\beta) = \frac{a(\alpha+\beta)}{a(\alpha)a(\beta)} \; , \quad a(\beta) = \frac{\Gamma(\beta')}{\pi^{D/2}\, 2^{2\beta}\, \Gamma(\beta)} \; .$$

We'll discuss the group-theoretical interpretation of this identity.

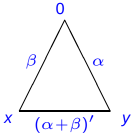
Graphic representation of Star Triangle Relation (reconstruction of Feynman graphs):



$= \frac{1}{(x-y)^{2\alpha}} \quad \Rightarrow$



$= G(\alpha, \beta) \cdot$



Operator representation of STR: (API, 2003)

$$\boxed{\hat{p}^{-2\alpha} \cdot \hat{q}^{-2(\alpha+\beta)} \cdot \hat{p}^{-2\beta} = \hat{q}^{-2\beta} \cdot \hat{p}^{-2(\alpha+\beta)} \cdot \hat{q}^{-2\alpha}} \;\; !!!$$

where we have used Heisenberg algebra:

$$[\hat{q}_\mu, \; \hat{p}_\nu] = \delta_{\mu\nu}$$

. Proof.

$$\langle x|\hat{p}^{-2\alpha} \cdot \hat{q}^{-2(\alpha+\beta)} \cdot \hat{p}^{-2\beta}|y\rangle = \langle x|\hat{q}^{-2\beta} \cdot \hat{p}^{-2(\alpha+\beta)} \cdot \hat{q}^{-2\alpha}|y\rangle$$

$$\hat{q}^{-2\alpha}|y\rangle = |y\rangle \, y^{-2\alpha} \, , \quad \langle x|\hat{p}^{-2\alpha}|y\rangle = a(\alpha)\,(x-y)^{-2\alpha'} \, .$$

Any STR is related to a solution $\mathcal{R}$ of the Yang-Baxter equation (YBE)

$$\mathcal{R}_{12}(u)\,\mathcal{R}_{23}(u+v)\,\mathcal{R}_{12}(v) = \mathcal{R}_{23}(v)\,\mathcal{R}_{12}(u+v)\,\mathcal{R}_{23}(u) \in \mathrm{End}(V\otimes V\otimes V)$$

u, v – are spectral parameters and $1, 2, 3$ are numbers of vect. spaces

YBE $\Longrightarrow$ Integrable Model

e.g. STR $\Longrightarrow$ Zamolodchikov's "Fishnet" diagram Int. Model.

Our aim is to find $\mathcal{R}$ which corresponds to the operator-type STR:

$$\hat{p}^{\,2u} \cdot \hat{q}^{2(u+v)} \cdot \hat{p}^{\,2v} = \hat{q}^{2v} \cdot \hat{p}^{\,2(u+v)} \cdot \hat{q}^{2u} \,. \qquad (1)$$

Consider two copies of the Heisenberg algebra $\{\hat{p}_1, \hat{q}_1\}$ and $\{\hat{p}_2, \hat{q}_2\}$:

$$[q_k^{\mu}, \hat{p}_j^{\nu}] = i\delta_{kj}\delta^{\mu\nu} \,.$$

Eq. (1) can be written in two equivalent forms $(1 \leftrightarrow 2)$

$$\hat{p}_2^{2u} \cdot \hat{q}_{12}^{2(u+v)} \cdot \hat{p}_2^{2v} = \hat{q}_{12}^{2v} \cdot \hat{p}_2^{2(u+v)} \cdot \hat{q}_{12}^{2v} \,,$$

$$\hat{p}_1^{2u} \cdot q_{12}^{2(u+v)} \cdot \hat{p}_1^{2v} = q_{12}^{2v} \cdot \hat{p}_1^{2(u+v)} \cdot \hat{q}_{12}^{2v} \,,$$

where $\hat{q}_{12}^{\mu} = \hat{q}_1^{\mu} - \hat{q}_2^{\mu}$.

Then, by using these two star-triangle identities, one can prove that $\mathcal{R}$-operator (D.Chicherin, API, S.Derkachov, 2012)

$$\underline{\mathcal{R}_{12}(u-v) = \hat{q}_{12}^{2(u_- - v_+)} \cdot \hat{p}_2^{2(u_+ - v_+)} \cdot \hat{p}_1^{2(u_- - v_-)} \cdot \hat{q}_{12}^{2(u_+ - v_-)}}$$

$$\in \mathrm{End}(V_{\Delta_1} \otimes V_{\Delta_2}) \; ,$$

where V_Δ is the space of conformal fields with conf. dimension Δ and

$$u_+ = u + \tfrac{\Delta_1 - D}{2} \, , \;\; u_- = u - \tfrac{\Delta_1}{2} \, , \;\; v_+ = v + \tfrac{\Delta_2 - D}{2} \; , \;\;\; v_- = v - \tfrac{\Delta_2}{2} \; ,$$

is a solution of the YB equation

$$\mathcal{R}_{12}(u)\, \mathcal{R}_{23}(u+v)\, \mathcal{R}_{12}(v) = \mathcal{R}_{23}(v)\, \mathcal{R}_{12}(u+v)\, \mathcal{R}_{23}(u)$$

$$\in \mathrm{End}(V_{\Delta_1} \otimes V_{\Delta_2} \otimes V_{\Delta_3}) \; ,$$

$$\mathcal{R}_{23}(u-v) = \hat{q}_{23}^{2(u_- - v_+)} \cdot \hat{p}_3^{2(u_+ - v_+)} \cdot \hat{p}_2^{2(u_- - v_-)} \cdot \hat{q}_{23}^{2(u_+ - v_-)} ,$$

For $\Delta_1 = \Delta_2 = \cdots = \Delta_N = \Delta$ we define the set of operators $\mathcal{R}_{ab} \in \mathrm{End}(V_{\Delta_1} \otimes V_{\Delta_2} \otimes \cdots V_{\Delta_N})$ which act nontrivially only in the spaces with numbers $a, b \in 1, 2, 3, \ldots, N$ and are defined as following

$$\mathcal{R}_{ab}(\alpha;\xi) := (\hat{q}_{(ab)})^{2(\alpha+\xi)}(\hat{p}_{(a)})^{2\alpha}(\hat{p}_{(b)})^{2\alpha}(\hat{q}_{(ab)})^{2(\alpha-\xi)} = \\ = 1 + \alpha\, h_{(ab)}(\xi) + \alpha^2 \ldots \,,$$

where $\alpha = u - v$, $\xi = \frac{D}{2} - \Delta$ and Hamiltonian densities $h_{(ab)}(x)$ are

$$h_{(ab)}(\xi) = 2\ \ln(\hat{q}_{(ab)})^2 + (\hat{q}_{(ab)})^{2\xi}\ \ln(\hat{p}^{\,2}_{(a)}\,\hat{p}^{\,2}_{(b)})\,(\hat{q}_{(ab)})^{-2\xi} = \\ = \hat{p}^{\,-2\xi}_{(a)}\ \ln(\hat{q}_{(ab)})^2\,\hat{p}^{\,2\xi}_{(a)} + \hat{p}^{\,-2\xi}_{(b)}\ \ln(\hat{q}_{(ab)})^2\,\hat{p}^{\,2\xi}_{(b)} + \ln(\hat{p}^{\,2}_{(a)}\,\hat{p}^{\,2}_{(b)})\,.$$

Using the standard procedure one can construct an integrable system with Hamiltonian

$$H(\xi) = \sum_{a=1}^{N-1} h_{(a,a+1)}(\xi)\,.$$

For $D = 1$ and $\xi = 1/2$ this Hamiltonian reproduces the Hamiltonian for the Lipatov's integrable model which is related to BFKL equation (QCD for high energy scattering).

These models are models of spin chains related to noncompact Lie algebras.

Conjecture. For general case $D > 1$ and $\xi \neq 1/2$ the spectrum of the Hamiltonian

$$H(\xi) = \sum_{a=1}^{N-1} \left(2 \ln(\hat{q}_{(aa+1)})^2 + (\hat{q}_{(aa+1)})^{2\xi} \ln(\hat{p}_{(a)}^{\,2} \, \hat{p}_{(a+1)}^{\,2}) \, (\hat{q}_{(aa+1)})^{-2\xi} \right) ,$$

will be the same as for the Lipatov's Hamiltonian. But degeneracy and wave functions will be different.

To summarize:
our aim is to construct explicit form of L-operator which solves RLL equations with operator type $\mathcal{R}$-operator:

$$\text{I.}\quad \mathcal{R}(u-v)\,\mathrm{L}^{\alpha}{}_{\beta}(u)\otimes\mathrm{L}^{\beta}{}_{\gamma}(v)=\mathrm{L}^{\alpha}{}_{\beta}(v)\otimes\mathrm{L}^{\beta}{}_{\gamma}(u)\,\mathcal{R}(u-v)$$

where elements $\mathrm{L}^{\alpha}{}_{\beta}(u)\in\mathcal{U}(\mathrm{conf})$ are operators in V_{Δ}, operator $\mathcal{R}(u-v)$ is intertwiner $V_{\Delta}\otimes V_{\Delta'}\to V_{\Delta'}\otimes V_{\Delta}$ and u and v are spectral parameters.
Further we also need another type of RLL equations:

$$\text{II.}\quad R^{\alpha_1\alpha_2}_{\beta_1\beta_2}(u-v)\,\mathrm{L}^{\beta_1}_{\gamma_1}(u)\,\mathrm{L}^{\beta_2}_{\gamma_2}(v)=\mathrm{L}^{\alpha_2}_{\beta_2}(v)\,\mathrm{L}^{\alpha_1}_{\beta_1}(u)\,R^{\beta_1\beta_2}_{\gamma_1\gamma_2}(u-v)\,,$$

where $R(u-v)$ is a numerical matrix which acts in $V\otimes V$.

Such L-operator which solves both type of RLL-relations are main building block for constructing (and solving) of quantum integrable systems.

Remark. Consider rational Zamolodchikov's R-matrix

$$R^{\alpha_1\alpha_2}_{\beta_1\beta_2}(u) = u(u+\frac{n}{2}-1)\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} + (u+\frac{n}{2}-1)\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1} - u\, g^{\alpha_1\alpha_2}\, g_{\beta_1\beta_2} \; ,$$

where g is the metric in $\mathbb{R}^{p,q}$. For this R-matrix the RLL relations of the **II**-type define the Yangian $Y(so(p+1,q+1))$. I.e. a solution $\mathrm{L}(u)$:

$$\mathrm{L}^{\alpha}{}_{\beta}(u) = \mathrm{I} + \sum_{k=1}^{\infty} u^{-k}\, (L^{(k)})^{\alpha}{}_{\beta} \; ,$$

of such type RLL relations is a generating function of infinite number of generators $(L^{(k)})^{\alpha}{}_{\beta}$ of the Yangian $Y(so(p+1,q+1))$. Note that $(L^{(1)})^{\alpha}{}_{\beta}$ are elements of $so(p+1,q+1) \subset Y(so(p+1,q+1))$.

One can search a solution of the RLL relations in the form

$$\mathrm{L}_{\rho}(u) = \mathrm{I} + \sum_{k=1}^{N} u^{-k}\, L^{(k)} \;\in \mathrm{End}(V \otimes V_{\rho}) \; .$$

This solution is called N-order evaluation of the the Yangian $Y(so(p+1,q+1))$. Evaluation representations ???

If $\mathrm{L}^{\alpha}{}_{\beta}(u)$ is N-order evaluation of the the Yangian $Y(so(p+1,q+1))$ (polynomial in u), then the operator (monodromy matrix)

$$T^{\alpha_1}_{\beta_n}(u) = \mathrm{L}^{\alpha_1}{}_{\beta_1}(u+\xi_1) \otimes \mathrm{L}^{\beta_1}{}_{\beta_2}(u+\xi_2) \otimes \cdots \mathrm{L}^{\beta_{n-1}}{}_{\beta_n}(u+\xi_n) \; ,$$

(here ξ_j – are anisotropy parameters) also solves RLL equation and defines new representation $T^{\alpha_1}_{\beta_n}(u)$ of the Yangian.

One more motivation:

There is a conjecture that MHV, NMHV, ... amplitudes for $N=4$ $D=4$ SYM theory, possess Yangian symmetry with respect to the action of $Y(su(2,2))$.

Chicherin, Derkachev and Kirschner (2013) have shown that this symmetry for n-points amplitude M_n can be formulated in the form of the condition

$$T^{\alpha_1}_{\beta_n}(u) \cdot M_n = \lambda(u) \delta^{\alpha_1}_{\beta_n} \cdot M_n \; ,$$

which is nothing but eigenvalue problem for n-th monodromy matrix $T^{\alpha_1}_{\beta_n}(u)$.

Our aim is to generalize this approach to the case of $osp(N|M)$-algebras.

Now we recall the definition of the Lie algebra $\mathrm{conf}(\mathbb{R}^{p,q})$.
$\mathbb{R}^{p,q}$ — pseudoeuclidean space with the metric

$$g_{\mu\nu} = \mathrm{diag}(\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q}) \, .$$

$\mathrm{conf}(\mathbb{R}^{p,q})$ — Lie algebra of the conformal group in $\mathbb{R}^{p,q}$ generated by $\{L_{\mu\nu}, P_\mu, K_\mu, D\}$ $(\mu,\nu = 0,1,\dots,p+q-1)$:

$$[L_{\mu\nu}\,,\,L_{\rho\sigma}] = i\,(g_{\nu\rho}\,L_{\mu\sigma} + g_{\mu\sigma}\,L_{\nu\rho} - g_{\mu\rho}\,L_{\nu\sigma} - g_{\nu\sigma}\,L_{\mu\rho})$$

$$[K_\rho\,,\,L_{\mu\nu}] = i\,(g_{\rho\mu}\,K_\nu - g_{\rho\nu}\,K_\mu)\,, \quad [P_\rho\,,\,L_{\mu\nu}] = i\,(g_{\rho\mu}\,P_\nu - g_{\rho\nu}\,P_\mu)\,,$$

$$[D\,,\,P_\mu] = i\,P_\mu\,, \quad [D\,,\,K_\mu] = -i\,K_\mu\,,$$

$$[K_\mu\,,\,P_\nu] = 2i\,(g_{\mu\nu}\,D - L_{\mu\nu})\,, \quad [P_\mu\,,\,P_\nu] = 0\,,$$

$$[K_\mu\,,\,K_\nu] = 0\,, \quad [L_{\mu\nu}\,,\,D] = 0\,.$$

$L_{\mu\nu}$ – generators for the rotation group $SO(p,q)$ in $\mathbb{R}^{p,q}$,
P_ν – shift generators in $\mathbb{R}^{p,q}$,
D – dilatation operator,
K_ν – conformal boost generators.

We have the well known isomorphism:

$$\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$$

and on generators it is formulated as

$$L_{\mu\nu} = M_{\mu\nu}\ , \quad K_\mu = M_{n,\mu} - M_{n+1,\mu}\ ,$$

$$P_\mu = M_{n,\mu} + M_{n+1,\mu}\ , \quad D = -M_{n,n+1}\ , \quad (n = p+q)\ ,$$

where M_{ab} $(a, b = 0, 1, \ldots, n+1)$ generate $so(p+1, q+1)$:

$$[M_{ab}, M_{dc}] = i(g_{bd}M_{ac} + g_{ac}M_{bd} - g_{ad}M_{bc} - g_{bc}M_{ad})\ ,$$

$$g_{ab} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}, 1, -1)\ .$$

The quadratic Casimir operator for $\mathrm{conf}(\mathbb{R}^{p,q})$ is

$$C_2 = \frac{1}{2}M_{ab}M^{ab} = \frac{1}{2}(L_{\mu\nu}L^{\mu\nu} + P_\mu K^\mu + K_\mu P^\mu) - D^2\ .$$

Consider the first-order evaluation of the Yangian $Y(so(p+1, q+1))$.

Proposition 1. *The L-operator of* $\text{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$*-type, which solves* RLL *equation (of Yangian type, or type* **II***), has the explicit form:*

$$\mathrm{L}(u) = \mathrm{I} + u^{-1}\frac{1}{2}T_s(M^{ab}) \otimes \rho(M_{ab}) \ \in \mathrm{End}(V \otimes V_\rho) \ .$$

where M_{ab} *are generators of* $so(p+1, q+1)$ *in the representation* ρ*:*

$$\rho(M_{ab}) = y_a \frac{\partial}{\partial y^b} - y_b \frac{\partial}{\partial y^a} \ ,$$

and T_s *is spinor matrix representation of* $conf(\mathbb{R}^{p,q}) = so(p+1, q+1)$*.*

Spinor reps T_s of $conf(\mathbb{R}^{p,q}) = so(p+1, q+1)$

Let $n = p + q = 2\nu(= D)$ be even integer and γ_μ $(\mu = 0, ..., n-1)$ be $2^{\frac{n}{2}}$-dimensional gamma-matrices in $\mathbb{R}^{p,q}$:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\, g_{\mu\nu}\, I \,,$$

$$\gamma_{n+1} \equiv \alpha\, \gamma_0 \cdot \gamma_1 \cdots \gamma_{n-1} \,, \quad \alpha^2 = (-1)^{q+n(n-1)/2} = (-1)^{q-\nu} \,,$$

where α is such that $\gamma_{n+1}^2 = I$. Using gamma-matrices γ_μ in $\mathbb{R}^{p,q}$ one can construct representation T_s of $\text{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$

$$T_s(L_{\mu\nu}) = \frac{i}{4}\, [\gamma_\mu, \gamma_\nu] \equiv \ell_{\mu\nu} \,, \quad T_s(K_\mu) = \gamma_\mu\, \frac{(1-\gamma_{n+1})}{2} \equiv k_\mu \,,$$
$$T_s(P_\mu) = \gamma_\mu\, \frac{(1+\gamma_{n+1})}{2} \equiv p_\mu \,, \quad T_s(D) = -\frac{i}{2}\, \gamma_{n+1} \equiv d \,.$$

where $P^\pm = \frac{(1 \pm \gamma_{n+1})}{2}$ are Weyl projectors.

Further we search the L-operator

$$\mathrm{L}^{(\Delta)}(u) = u\,\mathrm{I} + \frac{1}{2} T_s(M^{ab}) \otimes \rho_\Delta(M_{ab}) \; ,$$

where ρ_Δ — the standard differential representation of $conf(\mathbb{R}^{p,q})$ in space of fields with conf. weights Δ (G. Mack and A. Salam (1969))

$$\rho_\Delta(P_\mu) = -i\partial_{x_\mu} \equiv \hat{p}_\mu \; , \quad \rho_\Delta(D) = x^\mu \hat{p}_\mu - i\Delta \; ,$$

$$\rho_\Delta(K_\mu) = 2\,x^\nu\,(\hat{\ell}_{\nu\mu} + S_{\nu\mu}) + (x^\nu x_\nu)\hat{p}_\mu - 2i\,\Delta x_\mu \; ,$$

$$\rho_\Delta(L_{\mu\nu}) = \hat{\ell}_{\mu\nu} + S_{\mu\nu} \; , \quad \hat{\ell}_{\mu\nu} \equiv (x_\nu \hat{p}_\mu - x_\mu \hat{p}_\nu) \; ,$$

where $x_\mu \equiv \hat{q}_\mu$ are coordinates in $\mathbb{R}^{p,q}$, $S_{\mu\nu} = -S_{\nu\mu}$ are spin generators (with the same commutation relations as for $\hat{\ell}_{\mu\nu}$) and $[S_{\mu\nu}, x_\rho] = 0 = [S_{\mu\nu}, \hat{p}_\rho]$. For the quadratic Casimir operator we have:

$$\rho_\Delta(C_2) = \frac{1}{2}\left(S_{\mu\nu}\, S^{\mu\nu} - \hat{\ell}_{\mu\nu}\, \hat{\ell}^{\mu\nu} \right) + \Delta(\Delta - n) \; .$$

The representations ρ_Δ and $\rho_{n-\Delta}$ are contragradient to each other and in particular we have $\rho_\Delta(C_2) = \rho_{n-\Delta}(C_2)$.

Let the representation ρ_Δ acts in space of conformal spin-tensor fields of the type $(\ell, \dot{\ell})$. The action of spin generators $S_{\mu\nu}$ on such fields is

$$[S_{\mu\nu}\Phi]^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}} = (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_1}^{\ \ \alpha}\,\Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha\alpha_2\cdots\alpha_{2\ell}} + \cdots + (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_{2\ell}}^{\ \ \alpha}\,\Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell-1}\alpha} + \\ + (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_1}_{\ \ \dot\alpha}\,\Phi^{\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}} + \cdots + (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_{2\ell}}_{\ \ \dot\alpha}\,\Phi^{\dot\alpha_1\cdots\dot\alpha_{2\ell-1}\dot\alpha}_{\alpha_1\cdots\alpha_{2\ell}}\,.$$

For symmetric representations it is convenient to work with the generating functions

$$\Phi(x,\lambda,\tilde\lambda) = \Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}}(x)\,\lambda^{\alpha_1}\cdots\lambda^{\alpha_{2\ell}}\,\tilde\lambda_{\dot\alpha_1}\cdots\tilde\lambda_{\dot\alpha_{2\dot\ell}}\;,$$

where λ and $\tilde\lambda$ are auxiliary spinors and the action of $S_{\mu\nu}$ is given by differential operators (over spinors) $S_{\mu\nu} = \lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda + \tilde\lambda\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda}$:

$$[S_{\mu\nu}\Phi]\,(x,\lambda,\tilde\lambda) = \left[\lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda + \tilde\lambda\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda}\right]\Phi(x,\lambda,\tilde\lambda)\;,$$

where $\lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda = \lambda_\alpha\,(\boldsymbol{\sigma}_{\mu\nu})^\alpha_{\ \ \beta}\,\partial_{\lambda_\beta}$, $\tilde\lambda\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda} = \tilde\lambda^{\dot\alpha}\,(\bar{\boldsymbol{\sigma}}_{\mu\nu})_{\dot\alpha}^{\ \ \dot\beta}\,\partial_{\tilde\lambda^{\dot\beta}}$.

Consider $\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$-type operator (first order evaluation of the Yangian $Y(so(p+1, q+1))$):

$$\mathrm{L}^{(\Delta,\ell,\dot{\ell})}(u) \equiv \mathrm{L}^{(\Delta,\ell,\dot{\ell})}(u_+, u_-) = u\,\mathrm{I} + \frac{1}{2} T_s(M^{ab}) \otimes \rho_{\Delta,\ell,\dot{\ell}}(M_{ab}) \ ,$$

where T_s is the spinor representation and $\rho_{\Delta,\ell,\dot{\ell}}$ is the differential representation of the conformal algebra $so(p+1, q+1)$ which acts on the conformal spin-tensor fields $\Phi_{\Delta,\ell,\dot{\ell}}(x)$;

$$u_+ = u + \frac{\Delta - n}{2} \ , \quad u_- = u - \frac{\Delta}{2} \ , \quad n = p + q \ ,$$

We have used the expression for the ”polarized” Casimir operator $\frac{1}{2} T_s(M^{ab}) \otimes \rho_{\Delta,\ell,\dot{\ell}}(M_{ab})$ which was discussed in context of the differential representation of the conformal algebra.

Proposition 2.

For trivial representation $S_{\mu\nu} = 0$ and any dimension $n = p + q$ the operator $\mathrm{L}^{(\Delta)}(u_+, u_-)$ satisfies the *RLL* relation

$$\mathcal{R}_{23}(u-v)\,(\mathrm{L}_2^{(\Delta_1)})^{\alpha}_{\beta}(u)\,(\mathrm{L}_3^{(\Delta_2)})^{\beta}_{\gamma}(v) =$$
$$= (\mathrm{L}_2^{(\Delta_2)})^{\alpha}_{\beta}(v)\,(\mathrm{L}_3^{(\Delta_1)})^{\beta}_{\gamma}(u)\,\mathcal{R}_{23}(u-v) \quad \in End(V \otimes V_{\Delta_1} \otimes V_{\Delta_2}) \,,$$

with $\mathcal{R}$-operator $\in End(V_{\Delta_1} \otimes V_{\Delta_2})$

$$\mathcal{R}_{12}(u-v) = q_{12}^{2(u_- - v_+)} \cdot \hat{p}_2^{\,2(u_+ - v_+)} \cdot \hat{p}_1^{\,2(u_- - v_-)} \cdot q_{12}^{2(u_+ - v_-)} \,.$$

The operator $\mathrm{L}^{(\Delta)}(u_+, u_-) \equiv \mathrm{L}^{(\Delta)}(u)$ is also intertwined by the spinorial R-matrix which acts in $End(V \otimes V)$ where V – is the space of spinor representation .

Let Γ_a be $2^{\frac{n}{2}+1}$-dim. gamma-matrices in $\mathbb{R}^{p+1,q+1}$ $(n = p + q)$ which generate the Clifford algebra with the basis

$$\Gamma_{a_1 \dots a_k} = \frac{1}{k!} \sum_{s \in \mathcal{S}_k} (-1)^{\mathrm{p}(s)} \Gamma_{s(a_1)} \cdots \Gamma_{s(a_k)} \;\; (k \leq n+2) \, ,$$

where $\mathrm{p}(s)$ denote the parity of $s \in \mathcal{S}_k$. The $SO(p+1, q+1)$-invariant spinorial R-matrix is (it is necessary to take Weyl projection)

$$\mathrm{R}(u) = \sum_{k=0}^{n+2} \frac{\mathrm{R}_k(u)}{k!} \cdot \Gamma_{a_1 \dots a_k} \otimes \Gamma^{a_1 \dots a_k} \quad \in \quad \mathrm{End}(V \otimes V) \, ,$$

where V is the $2^{\frac{n}{2}+1}$-dimensional space of spinor representation T of $SO(p+1, q+1)$. To satisfy the Yang-Baxter equation the functions $\mathrm{R}_k(u)$ have to obey the recurrent relations (R.Shankar and E.Witten (1978), Al.B.Zamolodchikov (1981), M.Karowsky and H.Thun (1981))

$$\mathrm{R}_{k+2}(u) = -\frac{u+k}{u+n-k} \mathrm{R}_k(u) \, .$$

Proposition 3.

Consider two special cases:

- Dimension $n = p + q$ of the space $\mathbb{R}^{p,q}$ is arbitrary but representation ρ_Δ of $\text{conf}(\mathbb{R}^{p,q})$ is related to the trivial representation of spin $S_{\mu\nu} = 0$.
- The dimension of the space $\mathbb{R}^{p,q}$ is fixed by $n = p + q = 4$ and representation $\rho_{\Delta,\ell,\dot{\ell}}$ of $\text{conf}(\mathbb{R}^{p,q})$ corresponds to arbitrary spin $(\ell, \dot{\ell})$: $S_{\mu\nu} \neq 0$.

For these cases, the operator $\mathrm{L}^{(\Delta)}(u)$ satisfies the *RLL* relation

$$\text{III.} \quad \mathrm{R}_{12}(u-v)\,\mathrm{L}_1^{(\Delta)}(u)\,\mathrm{L}_2^{(\Delta)}(v) = \mathrm{L}_1^{(\Delta)}(v)\,\mathrm{L}_2^{(\Delta)}(u)\,\mathrm{R}_{12}(u-v)$$

with the spinorial R-matrix $\mathrm{R}_{12}(u) \in \mathrm{End}(V \otimes V)$, where V is the $2^{\frac{n}{2}}$-dimensional space of spinor representation T_s of $\text{conf}(\mathbb{R}^{p,q})$ and indices $1, 2$ are numbers of spaces V.

Proposition 4.

For any representation of spin $S_{\mu\nu}$ and $n = p + q = 4$ the operator $\mathrm{L}^{(\Delta,\ell,\dot{\ell})}(u)$ satisfies the *RLL* relation

$$\textbf{IV}. \quad \mathcal{R}_{12}(u-v)\cdot(\mathrm{L}_1^{(\Delta_1,\ell_1,\dot{\ell}_1)})^{\alpha}_{\beta}(u)\cdot(\mathrm{L}_2^{(\Delta_2,\ell_2,\dot{\ell}_2)})^{\beta}_{\gamma}(v) =$$

$$= (\mathrm{L}_1^{(\Delta_2,\ell_2,\dot{\ell}_2)})^{\alpha}_{\beta}(v)\cdot(\mathrm{L}_2^{(\Delta_1,\ell_1,\dot{\ell}_1)})^{\beta}_{\gamma}(u)\cdot\mathcal{R}_{12}(u-v) \in$$

$$\in End(V \otimes V_{\Delta_1,\ell_1,\dot{\ell}_1} \otimes V_{\Delta_2,\ell_2,\dot{\ell}_2}) \, ,$$

with special Yang-Baxter R-operator

$$[\mathcal{R}_{12}\,\Phi](x_1\,,\,\lambda_1\,,\,\tilde{\lambda}_1\,;x_2\,,\,\lambda_2\,,\,\tilde{\lambda}_2\,) =$$

$$= \int \frac{\mathrm{d}^4q\,\mathrm{d}^4k\,\mathrm{d}^4y\,\mathrm{d}^4z\;\; e^{i\,(q+k)\,x_{21}}\,e^{i\,k\,(y-z)}}{q^{2(u_- - v_+ + 2)}z^{2(u_+ - v_+ + 2)}y^{2(u_- - v_- + 2)}k^{2(u_+ - v_- + 2)}}\cdot \qquad (2)$$

$$\cdot\Phi(x_1 - y\,,\,\lambda_2\,\mathbf{z}\,\overline{\mathbf{k}},\,\tilde{\lambda}_2\,\overline{\mathbf{q}}\,\mathbf{y}\,;\,x_2 - z\,,\,\lambda_1\,\mathbf{q}\,\overline{\mathbf{z}},\,\tilde{\lambda}_1\,\overline{\mathbf{y}}\,\mathbf{k}\,)\,,$$

where we have used compact notation

$$\mathbf{x} = \sigma_\mu x^\mu/|x|\,, \quad \overline{\mathbf{x}} = \overline{\sigma}_\mu\,x^\mu/|x|\,.$$

Final Remarks:

Remark 1. Green function for two fields of the types $(\ell, \dot{\ell})$ and $(\dot{\ell}, \ell)$ in conformal field theory is well known

$$(\Phi(X), \Phi(Y)) = \frac{1}{(2\ell)!} \frac{1}{(2\dot{\ell})!} \frac{\left(\tilde{\lambda}\,(\overline{\mathbf{x}-\mathbf{y}})\,\eta\right)^{2\ell} \left(\lambda\,(\mathbf{x}-\mathbf{y})\,\tilde{\eta}\right)^{2\dot{\ell}}}{(x-y)^{2(4-\Delta)}} .$$

Here $X = x, \lambda, \tilde{\lambda}$ and for simplicity we use compact notation

$$\mathbf{x} = \sigma_\mu \frac{x^\mu}{|x|} \; ; \; \overline{\mathbf{x}} = \bar{\sigma}_\mu \frac{x^\mu}{|x|} \tag{3}$$

Remark 2. The integrable model of the type of Zamolodchikov's "Fishnet" diagram Integrable System for $\mathcal{R}$ given in (2) is not known.

Remark 3. Proposition 3 has been recently generalized (J.Fuksa, API, D.Karakhanyan, R.Kirschner) to the cases of *sp* and *osp* Lie (super)algebras.

1. D. Chicherin, S. Derkachov, A.P. Isaev, *Conformal group: R-matrix and star-triangle relation*, JHEP 1304 (2013) 020 (48 pp.) .

2. D. Chicherin, S. Derkachov, A.P. Isaev, *The spinorial R-matrix*, J.Phys. A46 (2013) 485201 (21 pp.)

3. A.P. Isaev, D. Karakhanyan, R. Kirschner, *Orthogonal and symplectic Yangians and Yang- Baxter R-operators*, Nucl.Phys. B904 (2016) 124-147

Group-theoretical meaning of Yang-Baxter $\mathcal{R}$-operator.
The Yang-Baxter $\mathcal{R}$-operator acts in the tensor product of two representation spaces of conformal algebra $\mathrm{conf}(\mathbb{R}^D) = so(D+1,1)$

$$\Phi_{\Delta_1}(x_1) \otimes \Phi_{\Delta_2}(x_2) \in V_{\Delta_1} \otimes V_{\Delta_2} \ ,$$

where $\Phi_\Delta(x)$ are spinless fields with conformal dimension Δ.
The meaning of $\mathcal{R}$: it intertwines two representations

$$\mathcal{R}_{12}(u-v): \quad V_{\Delta_1} \otimes V_{\Delta_2} \to V_{\Delta_2} \otimes V_{\Delta_1} \ .$$

or

$$\mathcal{R}_{12}(u-v) \cdot A_{\Delta_1} \otimes B_{\Delta_2} = B'_{\Delta_2} \otimes A'_{\Delta_1} \cdot \mathcal{R}_{12}(u-v) \ .$$

where $A_{\Delta_1}, A'_{\Delta_1} \in \mathrm{End}(V_{\Delta_1})$ and $B_{\Delta_2}, B'_{\Delta_2} \in \mathrm{End}(V_{\Delta_2})$.

To demonstrate this we construct L-operator (quantum analog of a Lax operator — important object in quantum integr. models)

$$||(\mathrm{L}^{(\Delta)})^{\alpha}_{\beta}|| = \mathrm{L}^{(\Delta)} \quad : \quad V \otimes V_{\Delta} \to V \otimes V_{\Delta}$$

where V is the space of a finite dim. (e.g., spinor) representation T_s of $\mathrm{conf}(\mathbb{R}^D)$. The L-operator is the operator which satisfies RLL relations

$$\mathcal{R}_{23}(u-v)\,(\mathrm{L}_2^{(\Delta_1)})^{\alpha}_{\beta}(u)\,(\mathrm{L}_3^{(\Delta_2)})^{\beta}_{\gamma}(v) = (\mathrm{L}_2^{(\Delta_2)})^{\alpha}_{\beta}(v)\,(\mathrm{L}_3^{(\Delta_1)})^{\beta}_{\gamma}(u)\,\mathcal{R}_{23}(u-v)\;,$$

where

$$(\mathrm{L}_2^{(\Delta)})^{\alpha}_{\beta} \;\in\; T_s(\mathcal{U}(\mathrm{conf}))^{\alpha}_{\beta} \otimes \rho_{\Delta}(\mathcal{U}(\mathrm{conf})) \otimes 1\;,$$
$$(\mathrm{L}_3^{(\Delta)})^{\alpha}_{\beta} \;\in\; T_s(\mathcal{U}(\mathrm{conf}))^{\alpha}_{\beta} \otimes 1 \otimes \rho_{\Delta}(\mathcal{U}(\mathrm{conf}))\;.$$

Here $\mathcal{U}(\mathrm{conf})$ is associative algebra (we specify it below), in particular it is enveloping algebra of $\mathrm{conf}(\mathbb{R}^{p,q}))$ and ρ_{Δ} is a differential representation of the conformal algebra which act in the space V_{Δ} of conformal fields.

Thus, in view of the *RLL* relations we should have

$$\mathcal{R}_{23}(u) \equiv I \otimes \mathcal{R}(u) \ : \ 1 \otimes \rho_{\Delta_1}(\mathcal{U}(\text{conf})) \otimes \rho_{\Delta_2}(\mathcal{U}(\text{conf})) \to \\ \to 1 \otimes \rho_{\Delta_2}(\mathcal{U}(\text{conf})) \otimes \rho_{\Delta_1}(\mathcal{U}(\text{conf}))$$

Further we will consider the general pseudoeuclidean space $\mathbb{R}^{p,q}$ $(p + q = D \Rightarrow n)$.

We choose the representation for γ_μ in $\mathbb{R}^{p,q}$ as:

$$\gamma_\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}_\mu \\ \overline{\boldsymbol{\sigma}}_\mu & \mathbf{0} \end{pmatrix} , \quad \gamma_{n+1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} ,$$

where $\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu + \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu = 2\, g_{\mu\nu} \mathbf{1}\,, \quad \overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu + \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu = 2\, g_{\mu\nu} \mathbf{1}\,.$
Thus, the representation T_s of $conf(\mathbb{R}^{p,q})$ is

$$\ell_{\mu\nu} = \begin{pmatrix} \frac{i}{4}(\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu - \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu) & \mathbf{0} \\ \mathbf{0} & \frac{i}{4}(\overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu - \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_{\mu\nu} & \mathbf{0} \\ \mathbf{0} & \overline{\boldsymbol{\sigma}}_{\mu\nu} \end{pmatrix} ,$$

$$p^\mu = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \overline{\boldsymbol{\sigma}}^\mu & \mathbf{0} \end{pmatrix} , \quad k^\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^\mu \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \quad d = -\frac{i}{2} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} .$$

Recall that

$$\boldsymbol{\sigma}_{\mu\nu} = ||(\boldsymbol{\sigma}_{\mu\nu})_\alpha{}^\beta|| \,, \quad \overline{\boldsymbol{\sigma}}_{\mu\nu} = ||(\overline{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}|| \,,$$

are inequivalent spinor representations of $so(p,q) = spin(p,q)$.

Any element of $conf(\mathbb{R}^{p,q})$ in the representation T_s is

$$A = i(\omega^{\mu\nu}\,\ell_{\mu\nu} + a^\mu\,p_\mu + b^\mu\,k_\mu + \beta\,d) =$$

$$= \begin{pmatrix} \frac{\beta}{2}\mathbf{1} + i\omega^{\mu\nu}\boldsymbol{\sigma}_{\mu\nu} & ib^\mu\boldsymbol{\sigma}_\mu \\ ia^\mu\overline{\boldsymbol{\sigma}}_\mu & -\frac{\beta}{2}\mathbf{1} + i\omega^{\mu\nu}\overline{\boldsymbol{\sigma}}_{\mu\nu} \end{pmatrix} \equiv \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} .$$

We consider A as the matrix of parameters $\omega^{\mu\nu}$, a^μ, b^μ, $\beta \in \mathbb{R}$.

Further we search the L-operator

$$\mathrm{L}^{(\Delta)}(u) = u\,\mathrm{I} + \frac{1}{2}T_s(M^{ab}) \otimes \rho_\Delta(M_{ab}) ,$$

where ρ_Δ — representation of $conf(\mathbb{R}^{p,q})$ on conformal fields with conf. weights Δ.

In representation ρ_Δ, the elements of $conf(\mathbb{R}^{p,q})$ act on the fields $\Phi(\mathbf{x})$:

$$\rho_\Delta(\omega^{\mu\nu} L_{\mu\nu} + a^\mu P_\mu + b^\mu K_\mu + \beta D)\,\Phi(\mathbf{x}) =$$
$$= \mathrm{Tr}_{T_s}\left[\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \big(T_s(M^{ab}) \cdot \rho(M_{ab})\big)\right] \Phi(\mathbf{x})\,.$$

where $\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}$ is the 2×2 block matrix of parameters, and the matrix of generators is

$$\frac{1}{2} T_s(M^{ab}) \cdot \rho_\Delta(M_{ab}) = (T_s \otimes \rho_\Delta)\left(\frac{1}{2} M^{ab} \otimes M_{ab}\right) =$$
$$= \begin{pmatrix} \frac{\Delta - n}{2} \cdot \mathbf{1} + \mathbf{S} - \mathbf{p} \cdot \mathbf{x}\,, & \mathbf{p} \\ \mathbf{x} \cdot \mathbf{S} - \overline{\mathbf{S}} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{p} \cdot \mathbf{x} + (\Delta - \frac{n}{2}) \cdot \mathbf{x}\,, & -\frac{\Delta}{2} \cdot \mathbf{1} + \overline{\mathbf{S}} + \mathbf{x} \cdot \mathbf{p} \end{pmatrix},$$

Here we introduced

$$\mathbf{p} = \tfrac{1}{2}\, \boldsymbol{\sigma}^\mu\, \hat{p}_\mu = -\tfrac{i}{2}\, \boldsymbol{\sigma}^\mu\, \partial_{x_\mu}\,, \quad \mathbf{x} = -i\, \overline{\boldsymbol{\sigma}}^\mu\, x_\mu\,,$$
$$\overline{\mathbf{S}} = \tfrac{1}{2}\, \overline{\boldsymbol{\sigma}}^{\mu\nu}\, S_{\mu\nu}\,, \quad \mathbf{S} = \tfrac{1}{2}\, \boldsymbol{\sigma}^{\mu\nu}\, S_{\mu\nu}\,.$$

For 4-dimensional case $\mathbb{R}^{p,q} = \mathbb{R}^{1,3}$ we have 2-component Weyl spinors $\lambda, \tilde{\lambda}$ and tensor fields $\Phi^{\dot{\alpha}_1 \cdots \dot{\alpha}_{2\dot{\ell}}}_{\alpha_1 \cdots \alpha_{2\ell}}(x)$ should be symmetric under permutations of dotted and undotted indices separately.
Then, for $n = 4$ we have

$$\boldsymbol{\sigma}_\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \,, \qquad \overline{\boldsymbol{\sigma}}_\mu = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3) \,,$$

where $\sigma_0 = \mathrm{I}_2$ and $\sigma_1, \sigma_2, \sigma_3$ are standard Pauli matrices.
Consequently we obtain for the self-dual components of $S_{\mu\nu}$

$$\mathbf{S} = \frac{1}{2} \boldsymbol{\sigma}^{\mu\nu} S_{\mu\nu} = \begin{pmatrix} \frac{1}{2} \lambda_1 \partial_{\lambda_1} - \frac{1}{2} \lambda_2 \partial_{\lambda_2} & \lambda_2 \partial_{\lambda_1} \\ \lambda_1 \partial_{\lambda_2} & -\frac{1}{2} \lambda_1 \partial_{\lambda_1} + \frac{1}{2} \lambda_2 \partial_{\lambda_2} \end{pmatrix}$$

and for anti-self-dual components of $S_{\mu\nu}$

$$\overline{\mathbf{S}} = \frac{1}{2} \overline{\boldsymbol{\sigma}}^{\mu\nu} S_{\mu\nu} = \begin{pmatrix} \frac{1}{2} \tilde{\lambda}^{\dot{1}} \partial_{\tilde{\lambda}^{\dot{1}}} - \frac{1}{2} \tilde{\lambda}^{\dot{2}} \partial_{\tilde{\lambda}^{\dot{2}}} & \tilde{\lambda}^{\dot{2}} \partial_{\tilde{\lambda}^{\dot{1}}} \\ \tilde{\lambda}^{\dot{1}} \partial_{\tilde{\lambda}^{\dot{2}}} & -\frac{1}{2} \tilde{\lambda}^{\dot{1}} \partial_{\tilde{\lambda}^{\dot{1}}} + \frac{1}{2} \tilde{\lambda}^{\dot{2}} \partial_{\tilde{\lambda}^{\dot{2}}} \end{pmatrix}$$