Ginzburg Centennial Conference on Physics, Lebedev Insitute, Moscow

Bianchi-I cosmological model and crossing singularities

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May 29 - June 3, 2017

Based on

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Transformations between Jordan and Einstein frames: Bounces, antigravity, and crossing singularities,

Physical Review D 94 (2016) 063510

Bianchi-I cosmological model and crossing singularities,

Physical Review D 95 (2017) 083503

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Introduction

- Cosmological singularities constitute one of the main problems of modern cosmology.
- The discovery of the cosmic acceleration stimulated the development of “exotic” cosmological models of dark energy; some of these models possess the so called soft or sudden singularities characterized by the finite value of the radius of the universe and its Hubble parameter.

- It is possible to cross the soft singularities.
- "Traditional" or "hard" singularities are associated with a zero volume of the universe (or of its scale factor), and with infinite values of the Hubble parameter, of the energy density and of the pressure –Big Bang and Big Crunch.

- Crossing of the Big Bang - Big Crunch singularity: is it a question of a field parametrization ?

- What happens in anisotropic spacetimes ?

Big Bang – Big Crunch crossing ?

- The idea that the Big Bang - Big Crunch singularity can be crossed appears very counterintuitive.
- Some approaches to the description of this crossing were elaborated during the last decade (I. Bars, S.H. Chen, P.J. Steinhardt and N. Turok, C. Wetterich, P. Dominis Prester).
- There is an analogy with the horizon which arises due to a certain choice of the spacetime coordinates: the singularity arises because of some choice of the field parametrization.

- On choosing some convenient field parametrization one can provide matching between the characteristics of the universe before and after the singularity crossing.
- Analogy with Kruskal coordinates for the Schwarzschild metric.
- On choosing appropriate combinations of the field variables we can describe the passage through the Big Bang - Big Crunch singularity, but this does not mean that the presence of such a singularity is not essential. Indeed, extended objects cannot survive this passage.

Friedmann cosmology in the presence of a scalar field: Einstein frame versus Jordan frame

$$S = \int d^4x\sqrt{-g}\left[U(\sigma)R - \frac{1}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} + V(\sigma)\right]$$

Conformal coupling

$$U(\sigma) = U_0 - \frac{1}{12}\sigma^2$$

A conformal transformation of the metric

$$g_{\mu\nu} = \frac{U_1}{U}\tilde{g}_{\mu\nu},$$

A new scalar field ϕ:

$$\frac{d\phi}{d\sigma} = \frac{\sqrt{U_1(U+3U'^2)}}{U} \quad \Rightarrow \quad \phi = \int \frac{\sqrt{U_1(U+3U'^2)}}{U} d\sigma.$$

$$\phi = \sqrt{3U_1}\ln\left[\frac{\sqrt{12U_0}+\sigma}{\sqrt{12U_0}-\sigma}\right]$$

$$\sigma = \sqrt{12U_0}\tanh\left[\frac{\phi}{\sqrt{12U_1}}\right].$$

The action then becomes the action for a minimally coupled scalar field:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[U_1 R(\tilde{g}) - \frac{1}{2}\tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + W(\phi) \right],$$

$$W(\phi) = \frac{U_1^2 V(\sigma(\phi))}{U^2(\sigma(\phi))}.$$

This is called the transformation from the Jordan frame to the Einstein frame.

In a flat Friedmann universe

$$ds^2 = N^2 d\tau^2 - a^2 dl^2,$$

$$d\tilde{s}^2 = \tilde{N}^2 d\tau^2 - \tilde{a}^2 dl^2.$$

$$\tilde{N} = \sqrt{\frac{U}{U_1}} N, \;\; \tilde{a} = \sqrt{\frac{U}{U_1}} a, \;\; t = \int \sqrt{\frac{U_1}{U}} d\tilde{t},$$

where t and $\tilde{t}$ are cosmic time parameters in the Jordan and the Einstein frames.

$$a = \tilde{a} \sqrt{\frac{U_1}{U_0}} \cosh\left(\frac{\phi}{\sqrt{12 U_1}}\right).$$

In the vicinity of the singularity in the Einstein frame:

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}} \to 0, \text{ when } \tilde{t} \to 0.$$

However, in the Jordan frame:

$$a \sim \tilde{t}^{\frac{1}{3}} \left(\tilde{t}^{\frac{1}{3}} + \tilde{t}^{-\frac{1}{3}} \right) \to \text{const} \neq 0.$$

Meanwhile, the scalar field σ crosses the value $\pm\sqrt{12U_0}$ and the coupling function U changes its sign.
Thus, the evolution in the Jordan frame is regular, and we can use this fact to describe the crossing of the Big Bang - Big Crunch singularity in the Einstein frame.

If one considers the expansion of the universe from the Big Bang with normal gravity driven by the standard scalar field, the continuation backward in time shows that it was preceded by the contraction towards a Big Crunch singularity in the antigravity regime, driven by a phantom scalar field with a negative kinetic term.

The possibility of a change of sign of the effective gravitational constant in the model with a conformaly coupled scalar field was analyzed in 1981 by A. Starobinsky, following the earlier suggestion made by A. Linde in 1980.

It was shown that in a homogeneous and isotropic universe, one can indeed cross the point where the effective gravitational constant changes sign. However, the presence of anisotropies changes the situation: these anisotropies grow indefinitely when this constant is equal to zero.

Bianchi-I universes filled with minimally coupled and conformally coupled massless scalar fields

$$d\tilde{s}^2 = \tilde{N}(\tau)^2 d\tau^2 - \tilde{a}^2(\tau)(e^{2\beta_1(\tau)}dx_1^2 + e^{2\beta_2(\tau)}dx_2^2 + e^{2\beta_3(\tau)}dx_3^2),$$

$$ds^2 = N(\tau)^2 d\tau^2 - a^2(\tau)(e^{2\beta_1(\tau)}dx_1^2 + e^{2\beta_2(\tau)}dx_2^2 + e^{2\beta_3(\tau)}dx_3^2),$$

$$\beta_1 + \beta_2 + \beta_3 = 0.$$

$$\dot{\beta}_i = \frac{\beta_{i0}}{\tilde{a}^3},\ \theta_0 = \beta_{10}^2 + \beta_{20}^2 + \beta_{30}^2.$$

$$\dot{\phi} = \frac{\phi_0}{\tilde{a}^3},\ \phi = \frac{\phi_0}{\left(\frac{3\theta_0}{2} + \frac{3\phi_0^2}{4U_1}\right)^{\frac{1}{2}}} \ln \tilde{t}.$$

$$\tilde{a}(\tilde{t}) = \left(\frac{3\theta_0}{2} + \frac{3\phi_0^2}{4U_1}\right)^{\frac{1}{6}} \tilde{t}^{1/3}.$$

We can represent the exact solution for the Bianchi-I universe in a standard Kasner-like form:

$$ds^2 = d\tilde{t}^2 - a_1^2 \tilde{t}^{2p_1} dx_1^2 - a_2^2 \tilde{t}^{2p_2} dx_2^2 - a_3^2 \tilde{t}^{2p_3} dx_3^2,$$

$$p_i = \frac{1}{3} + \frac{\beta_{i0}}{\sqrt{\frac{3\theta_0}{2} + \frac{3\psi_1^2}{4U_1}}}$$

$$p_1 + p_2 + p_3 = 1,$$

$$p_1^2 + p_2^2 + p_3^2 = \frac{1}{3} + \frac{\theta_0}{\frac{3\theta_0}{2} + \frac{3\psi_1^2}{4U_1}} < 1.$$

Scalar field, conformally coupled to gravity

Asymptotic behavior of the universe in the vicinity of the Big Bang singularity.

$$\sigma = -\sqrt{12U_0} + \sigma_1 t^\delta,$$

$$a = a_0 t^\eta,$$

$$0 < \delta < 1$$

$$\eta > 0.$$

$$\delta = 1 - 3\eta$$

$$\sigma_0 = a_0^3 \delta \sigma_1.$$

A Bianchi-I universe in the vicinity of the singularity also has a Kasner-like form of the metric with the Kasner indices

$$p_i = \eta + \frac{\bar{\beta}_{i0}\sqrt{6\eta(1-2\eta)}}{\sqrt{\bar{\theta}_0}}.$$

$$p_1 + p_2 + p_3 = 3\eta < 1$$

$$p_1^2 + p_2^2 + p_3^2 = 3\eta(2-3\eta).$$

We can establish a relation between this parameter η and the parameters describing the evolution of the universe in the Einstein frame.

$$\eta = \frac{1-3\gamma}{3(1-\gamma)} < \frac{1}{3}.$$

The late stage of the evolution, when $t \to \infty$.

$$\eta = \frac{1+3\gamma}{3(1+\gamma)} > \frac{1}{3}.$$

The possibiliity of having all three Kasner indices positive depends on the ratio

$$\frac{(p_1+p_2+p_2)^2}{p_1^2+p_2^2+p_3^2}.$$

The combination of three positive Kasner indices is possible if this ratio is greater than 1.

$$\frac{(p_1+p_2+p_2)^2}{p_1^2+p_2^2+p_3^2} = \frac{9\eta^2}{3\eta(2-3\eta)} > 1,$$

$$\eta > \frac{1}{3}.$$

This situation can be realized only for a late-time asymptotic regime. If we wish to have all three Kasner indices necessarily positive for a given value of the ratio, such a ratio should be greater than 2.

$$\frac{(p_1 + p_2 + p_2)^2}{p_1^2 + p_2^2 + p_3^2} = \frac{9\eta^2}{3\eta(2 - 3\eta)} > 2,$$

$$\eta > \frac{4}{9}.$$

We can say that the higher the value of the index η is the more isotropic is the solution.

Singularity crossing in a Bianchi - I universe

In the vicinity of the singularity in the Einstein frame

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}}.$$

In the Jordan frame

$$a \sim \tilde{t}^{\frac{1}{3}}(\tilde{\gamma} + \tilde{t}^{-\gamma}) \to 0,$$

because

$$\gamma = \frac{\phi_0}{3\sqrt{\phi_0^2 + 2\theta_0 U_1}} < \frac{1}{3}.$$

Thus, one encounters the Big Bang singularity also in the Jordan frame.

Mixing between geometrical and matter degrees of freedom and the singularity crossing

The Friedmann model with a massless scalar field can be described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2,$$

where

$$x = \frac{4\sqrt{U_1}}{\sqrt{3}}\tilde{a}^{\frac{3}{2}} \cosh \frac{\sqrt{3}}{4\sqrt{U_1}}\phi, \;\; y = \frac{4\sqrt{U_1}}{\sqrt{3}}\tilde{a}^{\frac{3}{2}} \sinh \frac{\sqrt{3}}{4\sqrt{U_1}}\phi,$$

and the Friedmann equation is

$$\dot{x}^2 - \dot{y}^2 = 0.$$

Inverting,

$$\tilde{a}^3 = \frac{3(x^2 - y^2)}{16U_1},$$

$$\phi = \frac{4\sqrt{U_1}}{\sqrt{3}} \mathrm{arctanh} \frac{x}{y}.$$

Initially

$$x > |y|.$$

For the solution

$$x = x_1\tilde{t} + x_0, \;\; y = y_1\tilde{t} + y_0, \;\; x_1^2 = y_1^2,$$

on choosing the constants as

$$x_0 = y_0 = A > 0, \;\; x_1 = -y_1 = B > 0,$$

we have

$$\tilde{a}^3 = \frac{3AB\tilde{t}}{4U_1}.$$

We can perform a continuation in the (x, y) plane, to $x < |y|$ or, in other words, to $\tilde{t} < 0$. Such a continuation implies an antigravity regime and the transition to the phantom scalar field, just as in more complicated schemes, discussed before.

How can we generalize these considerations to the case when the anistropy term is present ?

$$L = \frac{1}{2}\dot{r}^2 - \frac{1}{2}r^2(\dot{\varphi}^2 + \dot{\varphi}_1^2 + \dot{\varphi_2}^2),$$

$$\varphi_1 = \sqrt{\frac{3}{8}}\alpha_1, \ \varphi_2\sqrt{\frac{3}{8}}\alpha_2,$$

$$\beta_1 = \frac{1}{\sqrt{6}}\alpha_1 + \frac{1}{\sqrt{2}}\alpha_2, \ \beta_2 = \frac{1}{\sqrt{6}}\alpha_1 - \frac{1}{\sqrt{2}}\alpha_2, \ \beta_3 = -\frac{2}{\sqrt{6}}\alpha_1.$$

We can again consider the plane (x, y) such that

$$x = r\cosh\Phi,$$
$$y = r\sinh\Phi,$$

where a new hyperbolic angle Φ is defined as

$$\Phi = \int d\tilde{t}\sqrt{\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}^2}.$$

We have reduced a four-dimensional problem to the old two-dimensional one, by using the fact that the variables α_1, α_2 and ϕ enter into the equation of motion for the scale factor $\tilde{a}$ only through the squares of their time derivatives.

The behaviour of the scale factor before and after the crossing of the singularity can be matched by using the transition to the new coordinates x and y, which mix geometrical and scalar field variables in a particular way.

To describe the behaviour of the anisotropic factors it is enough to fix the constants β_{i0}.

Conclusions and discussion

- General relativity contains many surprises concerning relations between the matter and geometry. It is enough to take it seriously.
- There is no reason to be afraid of singularities!