Short historical notes

Dislocations: V. Volterra (1905)
K. Kondo (1952)
J. F. Nye (1953)
B. Bilby, R. Bullough, E. Smith (1955)
E. Kröner, I. Dzyaloshinskii, G. Volovik, M. Kléman, I. Kunin,
J. Madore, A. Kadić, D. Edelen, H. Kleinert,
E. Bezerra de Mello, F. Moraes, C. Malyshev, M. Lazar, ...

Disclinations: F. Frank (1958)
I. Dzyaloshinskii, G. Volovik (1978)
J. Hertz (1978)

Torsion: E. Cartan (1922)

Torsion in gravity: A. Einstein, E. Schrödinger, H. Weyl,
T. Kibble, D. Sciama, R. Finkelstein,
F.Hehl, P. von der Heyde, G. Kerlich, J. Nester,
Y. Ne’eman, J. Nitsch, J. McCrea, J. Mielke, Yu. Obukhov
M. Blagojecić, I. Nikolić, M. Vasilić, K. Hayashi, T. Shirafuji,
E. Sezgin, P. van Nieuwenhuizen, I. Shapiro, ...

Chern-Simons term in the Geometric Theory of Defects

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Katanaev, Volovich Ann. Phys. 216(1992)1; ibid. 271(1999)203
Katanaev Theor.Math.Phys.135(2003)733; ibid. 138(2004)163
Phisics – Uspekhi 48(2005)675.
arXiv 1705.07888 [math-ph]

Notation

$\mathbb{R}^3$ - continuous elastic media = Euclidean three-dimensional space

$x^i, y^i \quad i=1,2,3$ - Cartesian coordinates

δ_{ij} - Euclidean metric

$u^i(x)$ - displacement vector field

$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ - strain tensor

σ^{ij} - stress tensor

Elasticity theory of small deformations

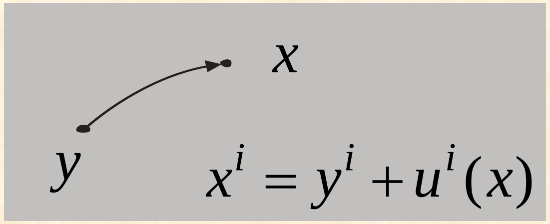
$\partial_i \sigma^{ij} + f^j = 0$ - Newton's law

$\sigma^{ij} = \lambda \delta^{ij} \varepsilon_k{}^k + 2\mu \varepsilon^{ij}$ - Hooke's law

$f^i(x)$ - density of nonelastic forces $(f^i = 0)$

λ, μ - Lame coefficients

Differential geometry of elastic deformations



$y^i \to x^i(y)$ - diffeomorphism: $\mathbb{R}^3 \to \mathbb{R}^3$

$$y^i \mapsto x^i$$

$$\delta_{ij} \qquad g_{ij}$$

$$g_{ij}(x) = \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\delta_{kl} \approx \delta_{ij} - \partial_i u_j - \partial_j u_i = \delta_{ij} - 2\varepsilon_{ij}$$ - induced metric $(*)$

$$\tilde{\Gamma}_{ijk} = \tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \neq 0$$ - Christoffel's symbols

$$\tilde{R}_{ijk}{}^{l} = \partial_i \tilde{\Gamma}_{jk}{}^{l} - \tilde{\Gamma}_{ik}{}^{m}\tilde{\Gamma}_{jm}{}^{l} - (i \leftrightarrow j) = 0$$ - curvature tensor

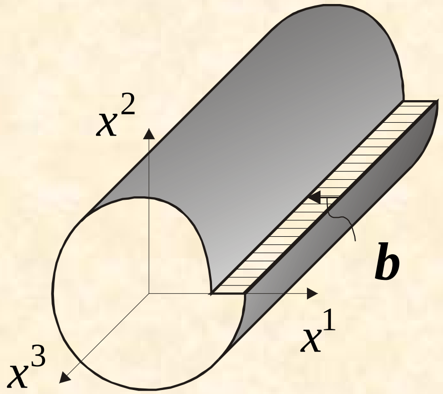
$$\ddot{x}^i = -\tilde{\Gamma}_{jk}{}^{i}\dot{x}^j\dot{x}^k$$ - extremals (geodesics)

$$R_{ijk}{}^{l} = 0$$ - Saint-Venant integrability conditions of $(*)$

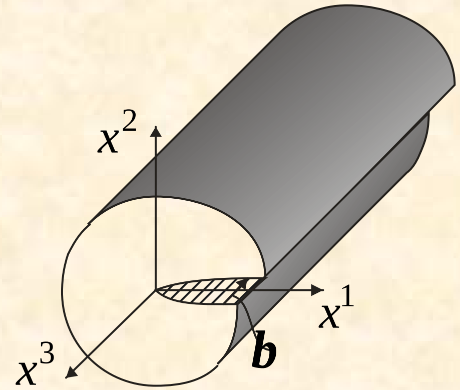
$$T_{ij}{}^{k} = \tilde{\Gamma}_{ij}{}^{k} - \tilde{\Gamma}_{ji}{}^{k} = 0$$ - torsion tensor

Dislocations

Linear defects:



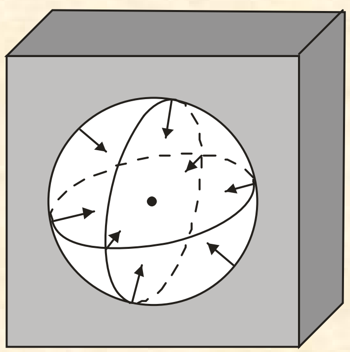
Edge dislocation



Screw dislocation

b - Burgers vector

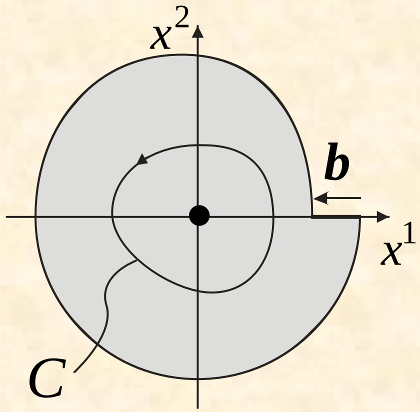
Point defects:



Vacancy

$$u^i(x) \begin{cases} \text{is continuous} & = \text{elastic deformations} \\ \text{is not continuous} & = \text{dislocations} \end{cases}$$

Edge dislocation



$$\oint_C dx^\mu \partial_\mu u^i = -\oint_C dx^\mu \partial_\mu y^i = -b^i \qquad (*)$$

$x^\mu,\ \mu = 1, 2, 3$ - arbitrary curvilinear coordinates

$y^i(x)$ - is not continuous !

$$e_\mu{}^i(x) = \begin{cases} \partial_\mu y^i & \text{- outside the cut} \\ \lim \partial_\mu y^i & \text{- on the cut} \end{cases}$$

- triad field (continuous on the cut)

$$(*) \Rightarrow b^i = \oint_C dx^\mu e_\mu{}^i = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i)$$ - Burgers vector in elasticity

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$ - torsion

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$ - curvature

$$\omega_\mu{}^{ij} = -\omega_\mu{}^{ji}$$

SO(3)-connection

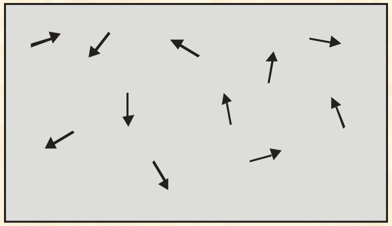
$$b^i = \iint dx^\mu \wedge dx^\nu T_{\mu\nu}{}^i$$

- definition of the Burgers vector in the geometric theory

Back to elasticity: if $R_{\mu\nu}{}^{ij} = 0$ then $\omega_\mu{}^{ij} \to 0$

Disclinations

Ferromagnets



$n^i(x)$ - unit vector field $\qquad n_0^i$ - fixed unit vector

$$n^i = n_0^j S_j{}^i(\omega)$$

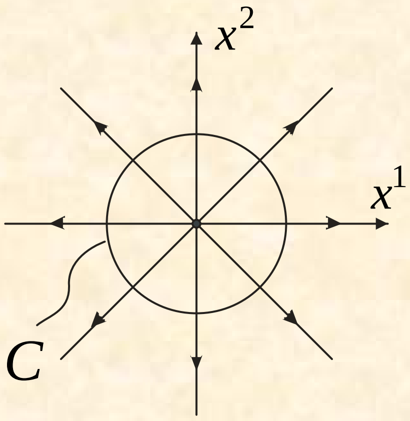
$S_i{}^j \in \mathbb{SO}(3)$ - orthogonal matrix

$\omega^{ij} = -\omega^{ji} \in \mathfrak{so}(3)$ - Lie algebra element (spin structure)

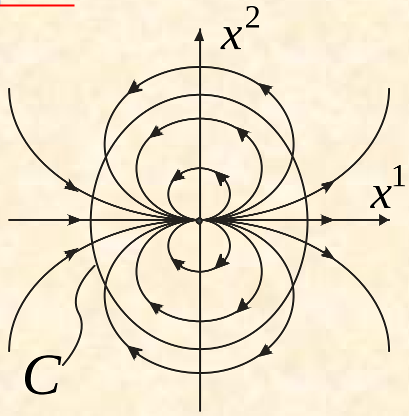
$\omega_i = \dfrac{1}{2}\varepsilon_{ijk}\omega^{jk}$ - rotational angle

ε_{ijk} - totally antisymmetric tensor $(\varepsilon_{123} = 1)$

Examples



$\Theta = 2\pi$



$\Theta = 4\pi$

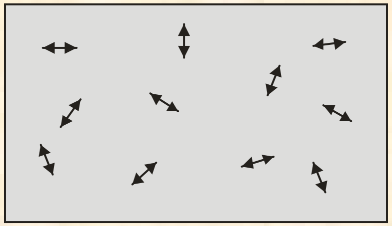
$$\Omega^{ij} = \oint_C dx^\mu \partial_\mu \omega^{ij}$$

$\Theta_i = \varepsilon_{ijk}\Omega^{jk}$ - Frank vector (total angle of rotation)

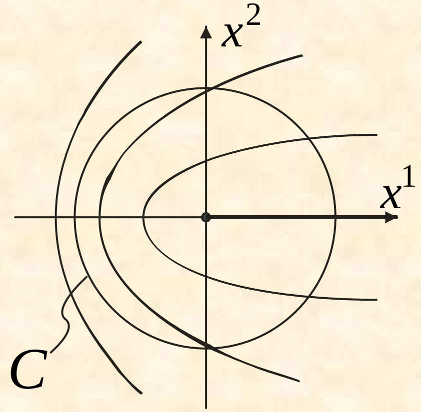
$$\Theta = \sqrt{\Theta^i \Theta_i}$$

More examples

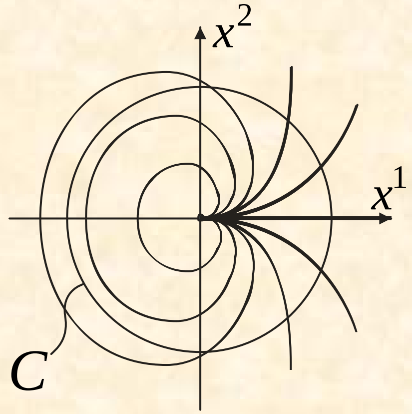
Nematic liquid crystals



$$n^i \sim -n^i$$



$\Theta = \pi$



$\Theta = 3\pi$

Model for a spin structure:

$\omega^i(x) \in \mathfrak{so}(3)$ - basic variable

$$S_i^{\ j} = \delta_i^j \cos\omega + \frac{\omega^k \varepsilon_{ki}^{\ \ j}}{\omega}\sin\omega + \frac{\omega_i \omega^j}{\omega^2}(1-\cos\omega) \in \mathbb{SO}(3), \qquad \omega = \sqrt{\omega^i \omega_i}$$

$l_{\mu i}^{\ \ j} = (\partial_\mu S^{-1})_i^{\ k} S_k^{\ j}$ - trivial SO(3)-connection (pure gauge)

$$\boxed{\partial^\mu l_\mu^{\ ij} = 0}$$ - principal chiral SO(3)-model

Frank vector

$\omega^{ij}(x)$ - is not continuous !

$$\omega_\mu{}^{ij}(x) = \begin{cases} \partial_\mu \omega^{ij} & \text{- outside the cut} \\ \lim \partial_\mu \omega^{ij} & \text{- on the cut} \end{cases}$$

- SO(3)-connection (continuous on the cut)

$$\Omega^{ij} = \oint dx^\mu \omega_\mu{}^{ij} = \iint dx^\mu \wedge dx^\nu (\partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij})$$ - the Frank vector

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$ - curvature

$$\Omega^{ij} = \iint dx^\mu \wedge dx^\nu R_{\mu\nu}{}^{ij}$$ - definition of the Frank vector in the geometric theory

Back to the spin structure: if $n \in \mathbb{R}^2$ then $\mathbb{SO}(3) \to \mathbb{SO}(2)$

Summary of the geometric approach

(physical interpretation)

Media with dislocations and disclinations $=$

$$= \mathbb{R}^3$$ with a given Riemann-Cartan geometry

Independent variables
$$\begin{cases} e_\mu{}^i & \text{- triad field} \\ \omega_\mu{}^{ij} & \text{- SO(3)-connection} \end{cases}$$

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$ - torsion (surface density of the Burgers vector)

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$ - curvature (surface density of the Frank vector)

Elastic deformations: $R_{\mu\nu}{}^{ij} = 0, \quad T_{\mu\nu}{}^i = 0$

Dislocations: $R_{\mu\nu}{}^{ij} = 0, \quad T_{\mu\nu}{}^i \neq 0$

Disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \quad T_{\mu\nu}{}^i = 0$

Dislocations and disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \quad T_{\mu\nu}{}^i \neq 0$

The free energy

$$S = S_{\mathrm{HE}}[e] + S_{\mathrm{CS}}[\omega]$$ - the total free energy

No dislocations: $g_{\mu\nu} = \delta_{\mu\nu} \quad \left(e_\mu{}^i = \delta_\mu^i\right)$ - Euclidean metric $\qquad S_{\mathrm{HE}} \to 0$

Three dimensions:

$$\omega_\mu{}^{ij} = \omega_{\mu k}\varepsilon^{kij}, \qquad \omega_{\mu k} := \tfrac{1}{2}\omega_\mu{}^{ij}\varepsilon_{ijk},$$ -SO(3) connection (1-form, the only variable)

$$\mu,\nu,\ldots = 1,2,3; \quad i,j,\ldots = 1,2,3$$ - indices

$$R_{\mu\nu k}(\omega) := R_{\mu\nu}{}^{ij}\varepsilon_{ijk} = 2\left(\partial_\mu\omega_{\nu k} - \partial_\nu\omega_{\mu k} + \omega_\mu{}^i\omega_\nu{}^j\varepsilon_{ijk}\right)$$ - curvature

$$S_{\mathrm{CS}} = \int_{\mathbb{R}^3}\left(\tfrac{1}{2}\omega^i \wedge d\omega_i + \tfrac{1}{3}\varepsilon_{ijk}\omega^i \wedge \omega^j \wedge \omega^k - \omega_i \wedge J^i\right)$$ - free energy for disclinations

J^i - the source term (2-form)

$$R_{\mu\nu}{}^k = J_{\mu\nu}{}^k$$ - equations of equilibrium

One linear disclination

$q^{\mu}(t) \in \mathbb{R}^3, \qquad t \in \mathbb{R}$ - the core of disclination

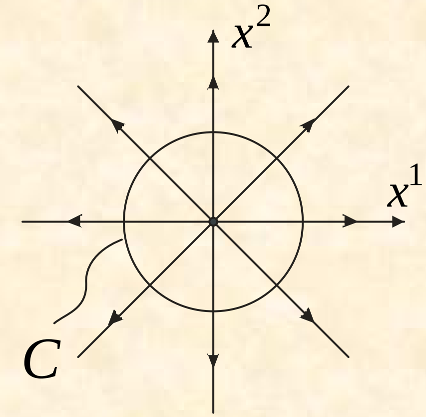
$S_{\text{int}} = \int dq^{\mu}\omega_{\mu i}J^{i} = \int dt\dot{q}^{\mu}\omega_{\mu i}J^{i} =$ - the interaction term

$$= \int dtd^3x\,\dot{q}^{\mu}\omega_{\mu i}J^{i}\delta^3(x-q) = \int d^3x\frac{\dot{q}^{\mu}}{\dot{q}^3}\omega_{\mu i}J^{i}\delta^2(\boldsymbol{x}-\boldsymbol{q})$$

$$\frac{\delta S_{\text{int}}}{\delta\omega_{\mu i}} = \frac{\dot{q}^{\mu}}{\dot{q}^3}J^{i}\delta^2(\boldsymbol{x}-\boldsymbol{q})$$

- the source term $\boldsymbol{x} = (x^1, x^2)$

One disclination along x^3 axis



Notation: $x^1 = x, \quad x^2 = y, \qquad z := x + iy$

$\omega_x{}^3,\ \omega_y{}^3$ - the only nontrivial components

$\omega_z{}^3 = \frac{1}{2}\omega_x{}^3 - \frac{i}{2}\omega_y{}^3$ - one complex component

$R_{z\bar{z}}{}^3 = 2\left(\partial_z\omega_{\bar{z}}{}^3 - \partial_{\bar{z}}\omega_z{}^3\right)$ - the curvature tensor

One straight linear disclination

Fixing the source term: $R_{z\bar{z}}{}^{3} = 4\pi i A\delta(z), \quad A \in \mathbb{R}$ - new kind of defect

The solution: $\omega_z{}^{3} = -\dfrac{iA}{z}$ $\qquad \partial_z \dfrac{1}{\bar{z}} = \pi\delta(z)$ - important formula

$$\omega_x{}^{3} = -\frac{2Ay}{x^2+y^2}, \qquad \omega_y{}^{3} = \frac{2Ax}{x^2+y^2}$$ - real components

Rotational angle field $\omega(\boldsymbol{x})$

$$\partial_x\omega = -\frac{2Ay}{x^2+y^2}, \qquad \partial_y\omega = \frac{2Ax}{x^2+y^2}$$

The integrability conditions $\partial_{xy}\omega = \partial_{yx}\omega$ are fulfilled

$$\omega = -2A\arctan\frac{x}{y} + C$$ - a general solution

$$C = \pi A \quad \Rightarrow \quad \tan\varphi = \frac{x}{y}, \quad \varphi := \frac{\omega}{2A}$$ - polar angle

One straight linear disclination

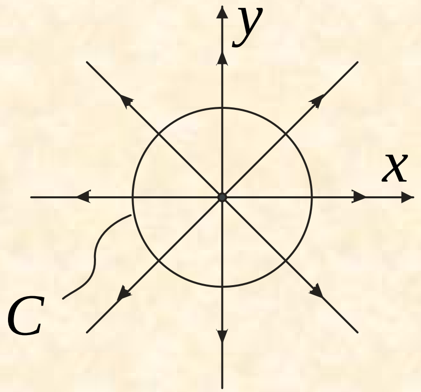
To make the rotational angle field $\omega(x, y)$ well defined, we must impose the quantization condition:

$$A = \frac{n}{2}, \quad n \in \mathbb{Z} \qquad \Rightarrow \qquad \omega = n\varphi$$

SO(3)-connection:

$$\omega_x^{12} = -\frac{ny}{x^2 + y^2} = -n \sin\varphi,$$

$$\omega_y^{12} = \frac{nx}{x^2 + y^2} = n \cos\varphi.$$



Conclusion

1) The first example of disclination is described within the geometric theory of defects.

2) The Chern-Simons term is well suited for disclinations in the geometric theory of defects.

3) Linear disclinations correspond to a new type of geometric singularity of the SO(3) connection.