Special values of coupling constants in Calogero models

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It is known (due to P. Etingof, I. Losev) that the algebra of observables (deformed Fock space) of the Calogero models based on the root system is simple for almost all values of the coupling constants. The algebra based on the root system $A_{N\text{-}1}$ is not simple iff $\nu = p/q$ with mutually prime p and q, $1 < q \leq N$ [I. Losev].

We consider the Calogero model based on rank-2 root systems and the bilinear forms generated by traces or by supertraces. We found all the values of the coupling constants for which some nonzero such forms become degenerate, and the algebra of observables acquires ideals, i.e., is not simple anymore.

Root system and Coxeter group

- Let $V = \mathbb{R}^N$
- For any $v \in V$ we define the reflection R_v :

$$R_v : \quad x \mapsto x - 2\frac{(v,x)}{(v,v)}v \quad \text{for any } x \in V$$

- Let $\mathcal{V} \subset V$ be unreduced root system, i.e.

for any $v,\ w \in \mathcal{V}$

a) $|\mathcal{V}| < \infty$,
b) $R_v \mathcal{V} = \mathcal{V}$,
c) $v = \alpha\ w \quad \Leftrightarrow \quad \alpha = \pm 1$

- Coxeter group $W(\mathcal{V})$ is the group generated by all the reflections R_v , $v \in \mathcal{V}$.
- Group algebra: $G(W) := \mathbb{C}[W(\mathcal{V})]$

Rational Cherednick algebra $\mathcal{H}(W,\nu_v)$ (algebra of observables)

- The Dunkl operators:

$$D_i := \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{v\in\mathcal{V}} \nu_v \frac{v_i}{(v,x)}(1-R_v)$$

- Generating elements (deformed creation and annihilation operators):

$$a_i^\alpha = \frac{1}{\sqrt{2}}\left(x_i + (-1)^\alpha D_i\right), \quad \alpha = 0,1$$

- Commutation relations:

$$R_v a_i^\alpha = \sum_{j=1}^{N}\left(\delta_{ij} - 2\frac{v_i v_j}{(v,v)}\right) a_j^\alpha R_v$$

$$[a_i^\alpha, a_j^\beta] = \varepsilon^{\alpha\beta}\left(\delta_{ij} + \sum_{v\in\mathcal{V}} \nu_v \frac{v_i v_j}{(v,v)} R_v\right)$$

- Parity: $\pi\,(a_i^{\alpha})=1$, $\pi(R_v)=0$

- The algebra $\mathcal{H}(W,\nu_v)$ is a (super)algebra of polynomials in a_i^{α} with coefficients in $G(W)$ (and parity π)

Calogero model

- Let ν be a central function on $W(\mathcal{V})$, $\quad g_v := \nu_v(\nu_v - 1)$
- Hamiltonian of Calogero model:

$$H_{Cal} := -\frac{1}{2}(\partial,\partial) + \frac{1}{2}(x,x) + \frac{1}{2}\sum_{v\in\mathcal{V}} g_v \frac{(v,v)}{(v,x)^2}.$$

- Let $\quad \beta := \prod_{v\in\mathcal{V}:\ (v,x)>0} (v,x)^{\nu_v}$

then $\quad H_\beta := \beta^{-1} H_{Cal}\beta = -\frac{1}{2}(\partial,\partial) + \frac{1}{2}(x,x) - \sum_{v\in\mathcal{V}^{(+)}} \frac{\nu_v}{(v,x)}(v,\partial)$

- Finally, $\quad H_\beta = T^{01} - \sum_{v\in\mathcal{V}^{(+)}} (\ldots)(1 - R_v)$

$$T^{01} = \frac{1}{2}\sum_i (a_i^0 a_i^1 + a_i^1 a_i^0)$$

sl_2

- Consider the elements

  $$T^{\alpha\beta} := 1/2 \sum_i \{a_i^{\alpha}, a_i^{\beta}\} \in \mathcal{H}(W, \nu)$$

  and the inner derivations they generate:

  $$\mathcal{D}^{\alpha\beta}:\ f \mapsto [f, T^{\alpha\beta}] \quad \text{for any} \quad f \in \mathcal{H}(W, \nu)$$

  They constitute the Lie algebra sl_2.

- Subalgebra of singlets

  $$\mathcal{H}_0(W, \nu) = \{ f \in \mathcal{H}(W, \nu) \mid [f, T^{\alpha\beta}] = 0 \}$$

- Calogero Hamiltonian:

  $$T^{01}|_{W\text{-symmetric space}} = H_\beta$$

- Fock procedure is based on:

$$[T^{01}, a_i^0] = -a_i^0, \qquad [T^{01}, a_i^1] = a_i^1$$

(Super)traces on $\mathcal{H}(W,\nu)$

- Trace: $tr(fg)=tr(gf)$
  supertrace: $str(fg)=(-1)^{\pi(f)\pi(g)}str(gf)$
- The numbers of traces $T(W)$ and supertraces $S(W)$ depends on W and satisfy inequality $T(W) \leq S(W)$.
- This inequality turns to equality $T(W) = S(W)$ if and only if W$\ni$ -1
- Let sp be some trace or supertrace on $\mathcal{H}(W,\nu)$.
  If $B_{sp}(f,g)=sp(fg)$, corresponding (super)symmetric bilinear form, is degenerate, then its null-vectors constitute a two-sided ideal in $\mathcal{H}(W,\nu)$.

Dihedral group $I_2(n)$

- We consider $\mathbb{R}^2$ as $\mathbb{C}$ with z and z^* as the basis elements
- Root system: $\mathcal{V}=\{v_1,\ldots,v_{2n}\}$, where

$$v_k = \exp\left(\frac{ik\pi}{n}\right)$$

- $I_2(n) \subset O(2,\mathbb{R})$
- Consists of n reflections R_k, and n rotations S_k, $k=1, \ldots n$:

$$R_k: \quad z \mapsto -z^* v_k^2, \quad z^* \mapsto -z v_k^{-2}$$

$$S_k: \quad z \mapsto v_k^{-2} z, \quad z^* \mapsto v_k^2 z^*$$

Property of $\mathcal{H}(I_2(n),\nu)$

- Let $\mathcal{S} := \frac{1}{2}(\{a_1^1, a_2^0\} - \{a_1^0, a_2^1\})$.
  This element generates $\mathcal{H}_0(I_2(n),\nu)$, namely
  $$\mathcal{H}_0(I_2(n),\nu) = G(I_2(n))[\mathcal{S}]$$
- $\mathcal{H}(I_2(2m+1),\nu)$ has m-dimensional space of traces and $(m+1)$-dimensional space of supertraces
- $\mathcal{H}(I_2(2m),\nu)$ has m-dimensional space of traces and isomorphic space of supertraces
- Let, for definiteness, n be odd.
- Let sp be some trace or supertrace on $\mathcal{H}(I_2(\mathrm{n}),\nu)$,
  B_{sp} be corresponding (super)symmetric bilinear form
  $$B_{sp}(f,g):=sp(fg),$$
  and $F_k^{sp}(t) := sp\left(\exp(t\mathcal{S} - it\nu \sum_i R_i)S_k\right)$

- **Theorem**. The bilinear form B_{sp} is degenerate if and only if the generating functions F_k^{sp} are integer for any k=1, ... , n.

Theorem. Let $m \in \mathbb{Z}$, $m \geqslant 1$ and $n=2m+1$.

- Then the associative algebra $\mathcal{H}(I_2(n),\nu)$

1) has a 1-parametric set of nonzero traces **tr** such that the symmetric invariant bilinear form $B_{\mathrm{tr}}(x,y)=\mathrm{tr}\,(xy)$ is degenerate, if

$$\nu = z/n, \text{ where } z \in \mathbb{Z}\backslash n\mathbb{Z},$$

2) has a 1-parametric set of nonzero supertraces **str** such that the supersymmetric invariant bilinear form $B_{\mathrm{str}}(x,y)=\mathrm{str}(xy)$ is degenerate, if

$$\nu = z/n, \qquad \text{where } z \in \mathbb{Z}\backslash n\mathbb{Z},$$

or $$\nu = z + 1/2\ , \quad \text{where } z \in \mathbb{Z}.$$

- For all other values of ν, all nonzero traces and supertraces are nondegenerate.

Theorem. Let $m \in \mathbb{Z}$, $m \geqslant 2$ and $n=2m$,
$\mu_0 = m(\nu_1+\nu_2)$, $\mu_1 = m(\nu_1 - \nu_2)$.

- Then the associative algebra $\mathcal{H}(I_2(n), \nu_1, \nu_2)$

1) has a 2-parametric set of nonzero traces tr such that the symmetric invariant bilinear form $B_{tr}(x,y)=tr\,(xy)$ is degenerate, if

$$\mu_0 \in \mathbb{Z} \backslash m\mathbb{Z}, \qquad \mu_1 \in \mathbb{Z} \backslash m\mathbb{Z},$$

2) has an 1-parametric set of nonzero traces tr such that the symmetric invariant bilinear form $B_{tr}(x,y)=tr\,(xy)$ is degenerate, if

$$\mu_0 \in \mathbb{Z} \backslash m\mathbb{Z}, \quad \mu_1 \notin \mathbb{Z} \backslash m\mathbb{Z}$$

or $\mu_1 \in \mathbb{Z} \backslash m\mathbb{Z}$, $\mu_0 \notin \mathbb{Z} \backslash m\mathbb{Z}$

or $\mu_0 \notin \mathbb{Z} \backslash m\mathbb{Z}$, $\mu_1 \notin \mathbb{Z} \backslash m\mathbb{Z}$ and $\mu_0 = \pm \mu_1 + m(2l+1)$.

- For all other values of ν, all nonzero traces are nondegenerate.