Hidden symmetries of deformed oscillators

Sergey Krivonos

Bogolyubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
Dubna, Russia

In collaboration with A. Sorin and O. Lechtenfeld

Main result

We associate with each simple Lie algebra a system of second-order differential equations invariant under a non-compact real form of the corresponding Lie group. In the limit of a contraction to a Schrödinger algebra, these equations reduce to a system of ordinary harmonic oscillators. We provide the clarifying example of such deformed oscillator: the system invariant under $G_{2(2)}$ transformations. The construction of invariant actions requires adding semi-dynamical degrees of freedom; we illustrate the algorithm with the example mentioned.

Plan

- Introduction
- $su(1,2)$ as deformation of Schrödinger algebra. Deformed oscillator
- 5-grading structure of the simple Lie algebras. General construction
- Oscillators with G_2 symmetry
- Conclusion

It is widely believed that integrability of a mechanical system is related with a high degree of (usually hidden) symmetry. Identifying such symmetry for a given system may be very complicated, even in the simplest cases, like in harmonic oscillators. The inverse task – constructing a system possessing a given symmetry – seems to be more simple, since there are many ways to find its equations of motion. One of them is the method of nonlinear realizations, equipped with the inverse Higgs phenomenon. For constructing a system of equations with a given symmetry, all one needs is the symmetry group together with the stability subgroup, which acts linearly on the mechanical coordinates.

Our recent paper arXiv:1607.03756 applies nonlinear realizations to the Schrödinger and ℓ-conformal Galilei algebra. These symmetries give rise to a system of ordinary harmonic oscillators and their higher-derivative (in time) extensions known as conformal Pais–Uhlenbeck oscillators. However, when we deform the Schrödinger algebra in two space dimensions to $su(1,2)$, the corresponding oscillator is also deformed to a nonlinear one. This suggests the existence of F-invariant nonlinearly deformed oscillator systems for every noncompact real Lie group F.

Crucial in our construction of the deformed oscillators is the 5-grading of $su(1,2)$. Now, any finite-dimensional simple complex Lie algebra beyond sl_2 has at least one non-compact real form with a 5-graded decomposition. A universal part of the 5-grading is the $su(1,1)$ sub-algebra formed by the highest- and lowest-grade subspace together with the (grade-zero) grading operator L_0, so one-dimensional conformal symmetry is always present. In the present talk, we will demonstrate how to extend this procedure from $su(1,2)$ to a non-compact real form of any simple Lie algebra. This procedure will provide a system of (generically nonlinear) second-order differential equations with the prescribed non-compact symmetry, which reduces to ordinary harmonic oscillators under the contraction to a Schrödinger algebra.

The simplest possibility to deform the Schrödinger algebra reads

$$\mathrm{i}\,[L_n, L_m] = (n-m)L_{n+m},\ \mathrm{i}\,[L_n, G_r] = \left(\frac{n}{2} - r\right) G_{n+r},\ \mathrm{i}\left[L_n, \overline{G}_r\right] = \left(\frac{n}{2} - r\right) \overline{G}_{n+r},$$
$$[U, G_r] = G_r, \quad \left[U, \overline{G}_r\right] = -\overline{G}_r,$$
$$\mathrm{i}\left[G_r, \overline{G}_s\right] = \gamma\left(\tfrac{3}{2}(r-s)U - \mathrm{i}L_{r+s}\right), \quad n,m = -1,0,1,\ r,s = -1/2, 1/2.$$

Here, γ is a deformation parameter: if $\gamma = 0$, we come back to the $\ell = \frac{1}{2}$ conformal Galilei algebra. The exact value of γ is inessential: if nonzero it can be put to unity by a rescaling of the generators G_r and $\overline{G}_r$.
We choose the stability subalgebra H as

$$H \ \propto \{L_0, L_1, U\}$$

and realize this deformed symmetry by left multiplications of

$$g = e^{\mathrm{i}t\left(L_{-1}+\omega^2 L_1\right)}\, e^{\mathrm{i}\left(uG_{-1/2}+\bar{u}\overline{G}_{-1/2}\right)}\, e^{\mathrm{i}\left(vG_{1/2}+\bar{v}\overline{G}_{1/2}\right)}.$$

$$g_0 = e^{\mathrm{i}\, aL_{-1}} : \qquad \begin{cases} \delta t = a\left(\sin^2(\omega t) + \frac{4\cos(2\omega t)}{4-\gamma^2\omega^2(u\,\bar u)^2}\right), \\ \delta u = -\frac{a}{2}\omega u\left(\sin(2\omega t) + \frac{4\mathrm{i}\,\gamma\omega\cos(2\omega t)}{4-\gamma^2\omega^2(u\,\bar u)^2}u\,\bar u\right), \end{cases}$$

$$g_0 = e^{\mathrm{i}\, bL_0} : \qquad \begin{cases} \delta t = \frac{b\sin(2\omega t)}{2\omega}\left(\frac{4+\gamma^2\omega^2(u\,\bar u)^2}{4-\gamma^2\omega^2(u\,\bar u)^2}\right), \\ \delta u = \frac{b}{2}u\left(\cos(2\omega t) - \frac{4\mathrm{i}\gamma\omega\sin(2\omega t)}{4-\gamma^2\omega^2(u\,\bar u)^2}u\,\bar u\right), \end{cases}$$

$$g_0 = e^{\mathrm{i}\, cL_1} : \qquad \begin{cases} \delta t = \frac{c}{\omega^2}\left(\cos^2(\omega t) - \frac{4\cos(2\omega t)}{4-\gamma^2\omega^2(u\,\bar u)^2}\right), \\ \delta u = \frac{c}{2\omega}u\left(\sin(2\omega t) + \frac{4\mathrm{i}\gamma\omega\cos(2\omega t)}{4-\gamma^2\omega^2(u\,\bar u)^2}u\,\bar u\right), \end{cases}$$

$$g_0 = e^{\mathrm{i}\left(aG_{-1/2}+\bar a\bar G_{-1/2}\right)} : \qquad \begin{cases} \delta t = \frac{2\mathrm{i}\gamma\cos(\omega t)(\bar a u - a\bar u)+\gamma^2\omega\sin(\omega t)(\bar a u + a\bar u)\,u\,\bar u}{4-\gamma^2\omega^2(u\,\bar u)^2} \\ \delta u = a\cos(\omega t) - \frac{\mathrm{i}\gamma\omega}{2}\sin(\omega t)\,u(2\bar a u + a\bar u) - \frac{\mathrm{i}}{2}\gamma\omega^2 u^2\,\bar u\,\delta t, \end{cases}$$

$$g_0 = e^{\mathrm{i}\left(bG_{1/2}+\bar b\bar G_{1/2}\right)} : \qquad \begin{cases} \delta t = \frac{2\mathrm{i}\gamma\sin(\omega t)\,(\bar b u - b\bar u)-\gamma^2\omega\cos(\omega t)(\bar b u + b\bar u)\,u\,\bar u}{\omega\,(4-\gamma^2\omega^2(u\,\bar u)^2)} \\ \delta u = \frac{\sin(\omega t)}{\omega}b + \frac{\mathrm{i}\gamma}{2}\cos(\omega t)\,u(2\bar b u + b\bar u) - \frac{\mathrm{i}}{2}\gamma\omega^2 u^2\,\bar u\,\delta t, \end{cases}$$

$$g_0 = e^{\mathrm{i}\alpha U} : \qquad \delta u = \mathrm{i}\alpha u.$$

In the limit $\gamma = 0$ they correctly reproduced the transformations preserving the action/equations of motion of the ordinary two-dimensional harmonic oscillator.

In what follows, we will need only the Cartan forms $\omega_{\pm 1/2}, \bar\omega_{\pm 1/2}$ and ω_U which read

$$
\begin{aligned}
\omega_{-1/2} &= du + \tfrac{i}{2}\gamma\,\omega^2 u^2\,\bar u\,dt - v d\tau, \quad \bar\omega_{-1/2} = \left(\omega_{-1/2}\right)^*, \\
\omega_{1/2} &= dv + \tfrac{i}{2}\gamma\, v^2\,\bar v\,d\tau - \tfrac{i}{2}\gamma\, v\,\left[2v\left(d\bar u - \tfrac{i}{2}\gamma\,\omega^2 u\,\bar u^2\,dt\right) + \bar v\left(du + \tfrac{i}{2}\gamma\,\omega^2 u^2\,\bar u\,dt\right)\right] \\
&\quad + \tfrac{3i}{2}\gamma\,\omega^2 v\,u\,\bar u\,dt + \omega^2\,u\,dt, \quad \bar\omega_{1/2} = \left(\omega_{1/2}\right)^*, \\
\omega_U &= \tfrac{3}{2}\gamma\left[v\,\bar v\,d\tau - v\left(d\bar u - \tfrac{i}{2}\gamma\,\omega^2 u\,\bar u^2\,dt\right) - \bar v\left(du + \tfrac{i}{2}\gamma\,\omega^2 u^2\,\bar u\,dt\right) + \omega^2 u\,\bar u\,dt\right],
\end{aligned}
$$

where

$$
d\tau = \left(1 + \tfrac{1}{4}\gamma^2\,\omega^2 u^2\,\bar u^2\right) dt + \tfrac{i}{2}\gamma\left(u\,d\bar u - \bar u\,du\right).
$$

The inverse Higgs constraints read

$$
\omega_{-1/2} = \bar\omega_{-1/2} = 0 \qquad \Rightarrow
$$

$$
v = \frac{\dot u + i\frac{\gamma\,\omega^2}{2} u^2\,\bar u}{1 + i\frac{\gamma}{2}\left(u\dot{\bar u} - \bar u\dot u\right) + \frac{\gamma^2\,\omega^2}{4} u^2\,\bar u^2}, \quad \bar v = \frac{\dot{\bar u} - i\frac{\gamma\,\omega^2}{2} u\,\bar u^2}{1 + i\frac{\gamma}{2}\left(u\dot{\bar u} - \bar u\dot u\right) + \frac{\gamma^2\,\omega^2}{4} u^2\,\bar u^2}.
$$

With above constraints taken into account, the form ω_U simplifies to

$$\omega_U = -\tfrac{3}{2}\gamma\left(v\,\bar{v}\,d\tau - \omega^2 u\,\bar{u}\,dt\right).$$

Observing that under all $SU(1,2)$ transformations the form ω_U only shifts by an exact differential, we can write down a simple invariant action,

$$S = -\tfrac{2}{3\gamma}\int \omega_U = \int dt\,\frac{\dot{u}\,\dot{\bar{u}} - \omega^2 u\,\bar{u}}{1 + \mathrm{i}\frac{\gamma}{2}(u\,\dot{\bar{u}} - \bar{u}\,\dot{u}) + \frac{1}{4}\gamma^2\omega^2 u^2\bar{u}^2}.$$

The equations of motion following from this action coincide with those obtained from the constraints

$$\omega_{1/2} = \bar{\omega}_{1/2} = 0 \qquad \Rightarrow \qquad \dot{v} - \mathrm{i}\gamma v^2\left(\dot{\bar{u}} - \tfrac{\mathrm{i}}{2}\gamma\,\omega^2 u\,\bar{u}^2\right) + \omega^2 u\left(\tfrac{3\mathrm{i}}{2}\gamma v\bar{u} + 1\right) = 0,$$

where $v, \bar{v}$ were defined above.

We conclude that the deformation of the symmetry algebra, i.e. the passing from the Schrödinger algebra to the $su(1,2)$ algebra produces a non-polynomial velocity dependence in the action. The “free”($\omega = 0$) system shares this feature. The undeformed ($\gamma = 0$) case describes a harmonic oscillator (or, with $\omega = 0$, a free particle).

The interesting questions is: which properties of the considered algebras (Schrödinger and $su(1,2)$ ones) are important for realization of the discussed procedure? The answer is simple: the crucial property is the existence of 5-grading decompositions of these algebras.

The 5-grading decomposition in the case of $su(1,2)$ algebra reads

$$\{L_{-1}\} \oplus \left\{G_{-\frac{1}{2}}, \overline{G}_{-\frac{1}{2}}\right\} \oplus \{L_0, U\} \oplus \left\{G_{\frac{1}{2}}, \overline{G}_{\frac{1}{2}}\right\} \oplus \{L_1\}\,.$$

This is a particular case of the general expression for 5-graded decomposition of Lie algebra $\mathcal{F}$ with respect to a suitable generator $L_0 \in \mathcal{F}$:

$$\mathcal{F} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_{-\frac{1}{2}} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_{+\frac{1}{2}} \oplus \mathfrak{f}_{+1} \qquad \text{with} \qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subseteq \mathfrak{f}_{i+j} \quad \text{for } i,j \in \left\{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\right\}$$

($\mathfrak{f}_i = 0$ for $|i| > 1$ understood).

- The grading is defined with respect to L_0: $[L_0, \mathfrak{f}_a] = -a\, \mathfrak{f}_a$
- The relations $\left[L_{-1}, \mathfrak{f}_{\frac{1}{2}}\right] \sim \mathfrak{f}_{-\frac{1}{2}}$ are crucial for the Inverse Higgs phenomenon, i.e. for the possibility to express the fields parameterized the space $\mathfrak{f}_{\frac{1}{2}}$ trough the fields parameterized the space $\mathfrak{f}_{-\frac{1}{2}}$
- The conditions $\omega_{\frac{1}{2}} = 0$ will always produce the second order equations of motion.

It is well known fact (B. Bina, M. Günaydin, Nucl. Phys. B **502** (1997) 713, `arXiv:hep-th/9703188`) that every simple Lie algebra $\mathcal{F}$ (except for sl_2) admits 5-graded decompositions with respect to a suitable generator $L_0 \in \mathcal{F}$:

$$\mathcal{F} = \mathfrak{f}_{-1} \oplus \mathfrak{f}_{-\frac{1}{2}} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_{+\frac{1}{2}} \oplus \mathfrak{f}_{+1} \qquad \text{with} \qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subseteq \mathfrak{f}_{i+j} \quad \text{for } i,j \in \left\{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\right\}$$

There is an (up to automorphisms) unique 5-grading with one-dimensional spaces $\mathfrak{f}_{\pm 1}$. Choosing this one, we may write

$$\mathfrak{f}_{-1} = \mathbb{C}\, L_{-1}, \qquad \mathfrak{f}_{+1} = \mathbb{C}\, L_1 \qquad \text{and} \qquad \mathfrak{f}_0 = \mathcal{H} \oplus \mathbb{C}\, L_0,$$

where $\mathcal{H} \subset \mathcal{F}$ is a Lie subalgebra and L_0 commutes with $\mathcal{H}$. A basis for the spaces $\mathfrak{f}_{\pm\frac{1}{2}}$ (of some dimension d) is given by generators $G^A_{\pm\frac{1}{2}}$ with $A = 1, \ldots, d$. They carry an irreducible representation of $\mathcal{H}$. In the following, we will deal with *real* Lie algebras and groups only, so some real form of $\mathcal{F}$ and $\mathcal{H}$ has to be picked. Compatibility with the 5-grading requires this real form to be non-compact. Therefore, (L_{-1}, L_1, L_0) generate an $su(1,1)$ subalgebra of $\mathcal{F}$. Different real forms of $\mathcal{F}$ and $\mathcal{H}$ give rise to different non-compact quaternionic symmetric spaces W

$$W = \frac{F}{H \times \mathrm{SU}(1,1)},$$

where F, H and SU(1,1) are the (simply-connected) groups generated by $\mathcal{F}$, $\mathcal{H}$ and $su(1,1)$, respectively.

Examples of the quaternionic symmetric spaces

$$\frac{F}{H \times \mathbf{SU}(1,1)}$$

- $F = SU(m,n),\ H = U(m-1,n-1)$ $\quad \dim(\mathfrak{f}_{-\frac{1}{2}}) = 2(m+n-2)$
- $F = SL(n,\mathbb{R}),\ H = GL(n-2,\mathbb{R})$ $\quad \dim(\mathfrak{f}_{-\frac{1}{2}}) = 2(n-2)$
- $F = SO(n,m),\ H = SO(n-2,m-2) \times SU(1,1)$ $\quad \dim(\mathfrak{f}_{-\frac{1}{2}}) = 2(m+n-4)$
- $F = SO(2n),\ H = SO(2n-4) \times SU(2)$ $\quad \dim(\mathfrak{f}_{-\frac{1}{2}}) = 4(n-2)$
- $F = G_{2(2)},\ H = SU(1,1)$ $\quad \dim(\mathfrak{f}_{-\frac{1}{2}}) = 4$

The main idea of our construction consists in enlarging the coset by slightly reducing the stability group from $H \times \mathrm{SU}(1,1)$ to $H \times \mathfrak{B}_{\mathrm{SU}(1,1)}$, where $\mathfrak{B}_{\mathrm{SU}(1,1)}$ denotes the positive Borel subgroup of SU(1,1), whose algebra $\mathfrak{b}_{su(1,1)}$ is generated by (L_0, L_1). In other words, we keep L_{-1} in the numerator and consider the coset

$$\mathcal{W} = \frac{F}{H \times \mathfrak{B}_{\mathrm{SU}(1,1)}}.$$

The elements of $\mathcal{W}$ can be parametrized as follows,

$$g = e^{t\left(L_{-1}+\omega^2 L_1\right)} e^{u(t)\cdot G_{-\frac{1}{2}}} e^{v(t)\cdot G_{\frac{1}{2}}},$$

where we employed a $\cdot$ notation to suppress the summation over A. The parameter ω represents some freedom in the parametrization of $\mathcal{W}$. It yields the oscillation frequency of the deformed oscillators we are going to construct.

Defining the Cartan forms in the standard way (with a basis $\{h_s\}$ of $\mathcal{H}$),

$$g^{-1}dg = \omega_{-1}L_{-1} + \omega_0 L_0 + \omega_1 L_1 + \omega_{-\frac{1}{2}} \cdot G_{-\frac{1}{2}} + \omega_{\frac{1}{2}} \cdot G_{\frac{1}{2}} + \sum_s \omega_h^s h_s ,$$

one can check that the constraints

$$\omega_{-\frac{1}{2}} = 0$$

firstly are invariant under the whole group F, realized by left multiplication in the coset $\mathcal{W}$, and secondly express the Goldstone fields $v(t)$ through the Goldstone fields $u(t)$ and their time derivatives in a covariant fashion (inverse Higgs phenomenon). After imposing these constraints we have a realization of the F transformations on the time t and the d coordinates $u_A(t)$.
Finally, one can impose the additional invariant constraints

$$\omega_{\frac{1}{2}} = 0,$$

which produces a system of second-order differential equations for the variables $u_A(t)$. These are the equations of motion.
Hence, with every simple Lie algebra $\mathcal{F}$ one may associate a system of dynamical equations in d variables which is invariant under some non-compact real form of the group F.

Comments

- Given the above structures, we can partially fix the commutator relations of $\mathcal{F}$:

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad \left[L_n, G^A_r\right] = \left(\tfrac{n}{2}-r\right) G^A_{n+r}$$

- The $[G, G]$ commutators lands in $\mathcal{H} \oplus su(1,1)$. However, they can be made to vanish by a group contraction. To this end, one rescales the generators via $G^A_{\pm\frac{1}{2}} = \gamma^{-1}\widetilde{G}^A_{\pm\frac{1}{2}}$ with $\gamma \in \mathbb{R}_+$. After the contraction $\gamma \to 0$ we arrive at the algebra

$$[L_n, L_m] = (n-m)L_{n+m}, \; \left[L_n, \widetilde{G}^A_r\right] = \left(\tfrac{n}{2}-r\right) \widetilde{G}^A_{n+r}, \; \left[\widetilde{G}^A_r, \widetilde{G}^B_s\right] = 0$$

  This is the Schrödinger algebra in $d+1$ dimensions.

- One may check that in this limit all equations linearize to

$$\ddot{u}_A(t) + \omega^2 u_A(t) = 0 \qquad \text{for} \quad A = 1, \ldots, d.$$

  Undoing the contraction, one may regard the corresponding equations as a deformation of the harmonic oscillators equations of motion. For this reason we refer to them as ‘deformed oscillators’.

- Finally we note that the above construction yields only the equations of motion for the variables $u_A(t)$. The question of existence of a corresponding invariant action has to be answered independently. We will demonstrate below that a positive answer requires extending further the number of Goldstone fields.

The 14-dimensional $g_{2(2)}$ algebra possesses a 5-grading with $\dim(\mathfrak{f}_{-\frac{1}{2}}) = 4$ and again $\mathcal{H} = su(1,1)$. This is made manifest by its commutation relations,

$$
\begin{aligned}
&[L_n, L_m] = (n-m)\,L_{n+m}, \quad [M_a, M_b] = (a-b)\,M_{a+b},\\
&[L_n, G_{r,A}] = \left(\tfrac{n}{2}-r\right) G_{n+r,A}, \quad [M_a, G_{r,A}] = \left(\tfrac{3a}{2}-A\right) G_{r,a+A},\\
&[G_{r,A}, G_{s,B}] = 3A\left(4A^2-5\right)\delta_{A+B,0}\,L_{r+s} + r\left(6A^2-8A\,B+6B^2-9\right)\delta_{r+s,0}\,M_{A+B},\\
&m,n=-1,0,1, \quad a,b=-1,0,1, \quad r,s=-\tfrac{1}{2},\tfrac{1}{2}, \quad A,B=-\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2}.
\end{aligned}
$$

Thus we have as basis elements

$$
G_{-\frac{1}{2},A} \in \mathfrak{f}_{-\frac{1}{2}}, \qquad G_{+\frac{1}{2},A} \in \mathfrak{f}_{+\frac{1}{2}} \qquad \text{and} \qquad M_a \in \mathcal{H} = su(1,1).
$$

We start from the eight-dimensional quaternionic symmetric space $W = G_{2(2)}/\mathrm{SO}(2,2)$ and enlarge it to the nine-dimensional coset

$$
\mathcal{W} = \frac{G_{2(2)}}{\mathrm{SU}(1,1)\times\mathfrak{B}_{\mathrm{SU}(1,1)}}
$$

with the stability subgroup generated by (L_0, L_1, M_a) as before. It may be parameterized as

$$
\begin{aligned}
g \;=\; & e^{t(L_{-1}+\omega^2 L_1)} e^{u_1 G_{-\frac{1}{2},-\frac{3}{2}} + u_2 G_{-\frac{1}{2},-\frac{1}{2}} + u_3 G_{-\frac{1}{2},+\frac{1}{2}} + u_4 G_{-\frac{1}{2},+\frac{3}{2}}}\\
& e^{v_1 G_{+\frac{1}{2},-\frac{3}{2}} + v_2 G_{+\frac{1}{2},-\frac{1}{2}} + v_3 G_{+\frac{1}{2},+\frac{1}{2}} + v_4 G_{+\frac{1}{2},+\frac{3}{2}}}, \quad g^\dagger = g^{-1}.
\end{aligned}
$$

The corresponding Cartan forms are rather complicated. To write them in a concise form we re-label the generators G and variables u and v in the spin-$\frac{3}{2}$ $\mathcal{H}$-representation with a symmetrized triple of spinor indices $\alpha,\beta,\gamma=1,2$:

$$G_{\pm\frac{1}{2},-\frac{3}{2}}=3G_{\pm\frac{1}{2},111},\,G_{\pm\frac{1}{2},-\frac{1}{2}}=3G_{\pm\frac{1}{2},112},\,G_{\pm\frac{1}{2},+\frac{1}{2}}=3G_{\pm\frac{1}{2},122},\,G_{\pm\frac{1}{2},+\frac{3}{2}}=3G_{\pm\frac{1}{2},222},$$
$$u_1=\tfrac{1}{3}U^{111},\quad u_2=U^{112},\quad u_3=U^{122},\quad u_4=\tfrac{1}{3}U^{222},$$
$$v_1=\tfrac{1}{3}V^{111},\quad v_2=V^{112},\quad v_3=V^{122},\quad v_4=\tfrac{1}{3}V^{222},$$

such that (with spinor index triples completely symmetric)

$$u_1G_{-\frac{1}{2},-\frac{3}{2}}+u_2G_{-\frac{1}{2},-\frac{1}{2}}+u_3G_{-\frac{1}{2},+\frac{1}{2}}+u_4G_{-\frac{1}{2},+\frac{3}{2}}=\sum_{\alpha\beta\gamma}U^{\alpha\beta\gamma}G_{-\frac{1}{2},\alpha\beta\gamma},$$
$$v_1\,G_{+\frac{1}{2},-\frac{3}{2}}+v_2\,G_{+\frac{1}{2},-\frac{1}{2}}+v_3\,G_{+\frac{1}{2},+\frac{1}{2}}+v_4\,G_{+\frac{1}{2},+\frac{3}{2}}=\sum_{\alpha\beta\gamma}V^{\alpha\beta\gamma}G_{+\frac{1}{2},\alpha\beta\gamma}.$$

Clearly, $G_{\pm\frac{1}{2},\alpha\beta\gamma}, U^{\alpha\beta\gamma}$, and $V^{\alpha\beta\gamma}$ are real tensors totally symmetric in α,β,γ.

Defining the Cartan forms

$$g^{-1}dg = \sum_n \omega_{L_n} L_n + \sum_a \omega_{M_a} M_a + \sum_{\alpha\beta\gamma} \omega_u^{\alpha\beta\gamma} G_{-\frac{1}{2},\alpha\beta\gamma} + \sum_{\alpha\beta\gamma} \omega_v^{\alpha\beta\gamma} G_{+\frac{1}{2},\alpha\beta\gamma},$$

we arrive at

$$\begin{aligned}
\omega_u^{\alpha\beta\gamma} &= dU^{\alpha\beta\gamma} + \omega^2 dt \left(U^3\right)^{\alpha\beta\gamma} - V^{\alpha\beta\gamma}\left[dt\left(1 - \frac{\omega^2}{2}\left(U^4\right)\right) + (UdU)\right],\\
\omega_v^{\alpha\beta\gamma} &= dV^{\alpha\beta\gamma} + (V^3)^{\alpha\beta\gamma}\left[dt\left(1 - \frac{\omega^2}{2}\left(U^4\right)\right) + (UdU)\right] - 2(VdUV)^{\alpha\beta\gamma} - (VVdU)^{\alpha\beta\gamma}\\
&\quad + \omega^2 dt\left[U^{\alpha\beta\gamma} + 3(UUV)^{\alpha\beta\gamma} + 2(VU^3V)^{\alpha\beta\gamma} - (U^3VV)^{\alpha\beta\gamma}\right].
\end{aligned}$$

In what follows we also need the forms ω_{M_a}

$$\omega_{M_{-1}} = \tfrac{1}{2}\omega^{11}, \qquad \omega_{M_{+1}} = \tfrac{1}{2}\omega^{22}, \qquad \omega_{M_0} = \omega^{12},$$

where

$$\omega^{\alpha\beta} = -4(VdU)^{\alpha\beta} + 2(VV)^{\alpha\beta}\left[dt\left(1 - \frac{\omega^2}{2}(U^4)\right) + (UdU)\right] + 2\omega^2 dt\left[(UU)^{\alpha\beta} + (U^3V)^{\alpha\beta}\right].$$

Now, imposing the conditions $\omega_u^{\alpha\beta\gamma} = 0$ we can express the coordinates $V^{\alpha\beta\gamma}$ in terms of $U^{\alpha\beta\gamma}$,

$$\omega_u^{\alpha\beta\gamma} = 0 \qquad \Rightarrow \qquad V^{\alpha\beta\gamma} = \frac{\dot{U}^{\alpha\beta\gamma} + \omega^2 \left(U^3\right)^{\alpha\beta\gamma}}{1 - \frac{\omega^2}{2}(U^4) + (U\dot{U})}.$$

Finally, using the conditions $\omega_v^{\alpha\beta\gamma} = 0$ we come to the covariant equations of motion (with $V = V(U)$):

$$\dot{V}^{\alpha\beta\gamma} + (V^3)^{\alpha\beta\gamma}\left[\left(1 - \frac{\omega^2}{2}\left(U^4\right)\right) + (U\dot{U})\right] - 2(V\dot{U}V)^{\alpha\beta\gamma} - (VV\dot{U})^{\alpha\beta\gamma}$$
$$+ \omega^2\left[U^{\alpha\beta\gamma} + 3(UUV)^{\alpha\beta\gamma} + 2(VU^3V)^{\alpha\beta\gamma} - (U^3VV)^{\alpha\beta\gamma}\right] = 0.$$

In the limit $\omega = 0$ these equations simplify to

$$\ddot{U}^{\alpha\beta\gamma} = 2\frac{(\dot{U}\dot{U}\dot{U})^{\alpha\beta\gamma} - \dot{U}^{\alpha\beta\gamma}(\dot{U}\dot{U}\dot{U}\cdot U)}{1 + (U\dot{U})} \quad \text{with} \quad (\dot{U}\dot{U}\dot{U}\cdot U) \equiv \sum (\dot{U}\dot{U}\dot{U})^{\alpha_1\alpha_2\alpha_3} U_{\alpha_1\alpha_2\alpha_3},$$

and in the contraction limit $\gamma \to 0$ after the rescaling $G^A_{\pm\frac{1}{2}} = \gamma^{-1}\widetilde{G}^A_{\pm\frac{1}{2}}$ they linearize to

$$\ddot{U}^{\alpha\beta\gamma} + \omega^2 U^{\alpha\beta\gamma} = 0.$$

In order to construct the invariant action one has to extend the coset to an eleven-dimensional one,

$$\mathcal{W} = \frac{G_{2(2)}}{\mathrm{SU}(1,1) \times \mathfrak{B}_{\mathrm{SU}(1,1)}} \qquad \rightarrow \qquad \mathcal{W}_{\mathrm{imp}} = \frac{G_{2(2)}}{\mathrm{U}(1) \times \mathfrak{B}_{\mathrm{SU}(1,1)}},$$

with elements

$$g_{\mathrm{imp}} = g\, e^{\Lambda_{-1} M_{-1} + \Lambda_{+1} M_{+1}}.$$

Finally, the invariant action can be constructed from Ω_{M_0},

$$S = -\int \Omega_{M_0} = -\int \frac{1}{1+\lambda_{-1}\lambda_{+1}} \left[\lambda_{-1}\tilde{\omega}^{22} - \lambda_{+1}\tilde{\omega}^{11} + (1-\lambda_{-1}\lambda_{+1})\,\tilde{\omega}^{12} + \lambda_{-1} d\lambda_{+1} - \lambda_{+1} d\lambda_{-1} \right],$$

where

$$\tilde{\omega}^{\alpha\beta} = -2dt \frac{(\dot{U}\dot{U})^{\alpha\beta} - \omega^2 \left[\left(1 + (U\dot{U})\right)(UU)^{\alpha\beta} - (U^3\dot{U})^{\alpha\beta} \right]}{1 - \frac{\omega^2}{2}(U^4) + (U\dot{U})}.$$

We proposed a procedure which associates with any simple Lie algebra a system of the second-order nonlinear differential equations which are invariant with respect to a non-compact real form of this symmetry. The explicit example considered in detail gives rise to a system of deformed oscillators invariant under $G_{2(2)}$ transformations. For this case, we also constructed invariant action. This action includes additional, semi-dynamical variables which do not affect the equations of motion for the physical variables.

The five-graded decomposition of the Lie algebra, a key feature in our construction, coercively includes a one-dimensional conformal algebra $su(1,1)$. Therefore, all systems constructed in this fashion will possess conformal invariance. Due to our special choice of the stability subalgebra a dilaton is absent, and the conformal invariance is achieved without it. In a contraction limit, when the Lie algebra reduces to a Schrödinger algebra, the equations reduce to a system of ordinary harmonic oscillators.

The following further developments come to mind.

- Our choice of the coset parametrization (the ordering $\mathfrak{g}_{-1} \cdot \mathfrak{g}_{-\frac{1}{2}} \cdot \mathfrak{g}_{\frac{1}{2}}$) is rather special. Clearly, this is far from unique, and a reordering will give the equations a different appearance.
- The chosen coset parametrization is computationally useful but provides an unusual form of the metric. It is desirable to bring the metric and connection to a more standard form through some reparametrization.
- Some Lie algebras possess other forms of grading (for example, there is a 7-graded basis for G_2). It will be interesting to learn how our equations change when the grading is altered.
- Our construction procedure for invariant actions works properly only in the presence of an $su(1,1)$ factor in the stability subalgebra. It should be clarified how to construct invariant actions when this is not so.
- A supersymmetric extension of the present approach may be of interest.
- Finally, a Hamiltonian description may illuminate the structure of conserved currents and help to relate our systems to others in the literature.