Superconformal higher spin multiplets

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SMK, R. Manvelyan & S. Theisen, "Off-shell superconformal higher spin multiplets in four dimensions," arXiv:1701.00682;
SMK & M. Tsulaia, "Off-shell massive N=1 supermultiplets in three dimensions," Nucl. Phys. B **914**, 160 (2017) arXiv:1609.06910;
SMK & D. Ogburn, "Off-shell higher spin N=2 supermultiplets in three dimensions," Phys. Rev. D **94**, 106010 (2016) arXiv:1603.04668.

Outline

General comments

- Conformal higher spin fields in four dimensions
  E. Fradkin & A. Tseytlin (1985)
- Gauging of the conformal higher spin superalgebras
  E. Fradkin & V. Linetsky (1989–1991)
- This talk is about the conformal higher spin superfields in four and three dimensions, which were introduced in 2016–17.

General comments

Conformal higher spin fields in four dimensions may naturally be introduced as by-products of Fronsdal's massless actions.

Massless spin-s fields:

$h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} \equiv h_{\alpha(s)\dot\alpha(s)}$ gauge field

$h_{\alpha_1\dots\alpha_{s-2}\dot\alpha_1\dots\dot\alpha_{s-2}}$ compensator

Gauge freedom

$$\delta h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = \partial_{(\alpha_1(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\dot\alpha_s)} ,$$

$$\delta h_{\alpha_1\dots\alpha_{s-2}\dot\alpha_1\dots\dot\alpha_{s-2}} = \partial^{\beta\dot\beta}\zeta_{\beta\alpha_1\dots\alpha_{s-2}\dot\beta\dot\alpha_1\dots\dot\alpha_{s-2}}$$

Conformal spin-s field: $h_{\alpha(s)\dot\alpha(s)}$ with the same gauge freedom conformal primary of dimensions $(2-s)$

Conformal primary and gauge-invariant field strength

$$\mathcal{C}_{\alpha_1\dots\alpha_{2s}} = \partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{(\alpha_s}{}^{\dot\beta_s} h_{\alpha_{s+1}\dots\alpha_{2s})\dot\beta_1\dots\dot\beta_s}$$

Conformal gauge invariant action

$$S = \int \mathrm{d}^4x\, \mathcal{C}^{\alpha_1\dots\alpha_{2s}}\mathcal{C}_{\alpha_1\dots\alpha_{2s}} + \mathrm{c.c.}$$

Superconformal transformations in $\mathbb{M}^{4|4}$

$\mathbb{M}^{4|4}$ Minkowski superspace with Cartesian coordinates $z^A = (x^a, \theta^\alpha, \bar\theta_{\dot\alpha})$
$D_A = (\partial_a, D_\alpha, \bar D^{\dot\alpha})$ superspace covariant derivatives

Superconformal transformations are generated by first-order operators

$$\xi = \xi^B D_B = \xi^b \partial_b + \xi^\beta D_\beta + \bar\xi_{\dot\beta} \bar D^{\dot\beta}$$

which obey the equation

$$\Big[\xi + \frac{1}{2} K^{bc}[\xi] M_{bc}\, , D_A\Big] + \delta_{\sigma[\xi]} D_A = 0 \ ,$$

for some local Lorentz ($K^{bc}[\xi]$) and super-Weyl ($\sigma[\xi]$) parameters.
ξ is called a conformal Killing supervector field.

The parameters ξ^α, $K^{bc}[\xi]$ and $\sigma[\xi]$ are expressed in terms of ξ^a:

$$\begin{aligned}
\xi^\alpha &= -\frac{\mathrm{i}}{8} \bar D_{\dot\alpha} \xi^{\dot\alpha\alpha} \ , \qquad && \bar D_{\dot\gamma} \xi^\alpha = 0 \ , \\
K_{\alpha\beta}[\xi] &= D_{(\alpha} \xi_{\beta)} \ , \qquad && \bar D_{\dot\gamma} K_{\alpha\beta}[\xi] = 0 \ , \\
\sigma[\xi] &= \frac{1}{3}(D_\alpha \xi^\alpha + 2 \bar D^{\dot\alpha} \bar\xi_{\dot\alpha}) \ , \qquad && \bar D_{\dot\gamma} \sigma[\xi] = 0 \ .
\end{aligned}$$

Super-Weyl transformations in supergravity

P. Howe & R. Tucker (1978)

$$
\begin{aligned}
\delta_\sigma \mathcal{D}_\alpha &= (\bar\sigma - \frac{1}{2}\sigma)\mathcal{D}_\alpha + (\mathcal{D}^\beta \sigma)\, M_{\alpha\beta} \ , \\
\delta_\sigma \bar{\mathcal{D}}_{\dot\alpha} &= (\sigma - \frac{1}{2}\bar\sigma)\bar{\mathcal{D}}_{\dot\alpha} + (\bar{\mathcal{D}}^{\dot\beta}\bar\sigma)\bar{M}_{\dot\alpha\dot\beta} \ , \\
\delta_\sigma \mathcal{D}_{\alpha\dot\alpha} &= \frac{1}{2}(\sigma+\bar\sigma)\mathcal{D}_{\alpha\dot\alpha} + \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}_{\dot\alpha}\bar\sigma)\mathcal{D}_\alpha + \frac{\mathrm{i}}{2}(\mathcal{D}_\alpha\sigma)\bar{\mathcal{D}}_{\dot\alpha} \\
&\quad +(\mathcal{D}^\beta{}_{\dot\alpha}\sigma)M_{\alpha\beta} + (\mathcal{D}_\alpha{}^{\dot\beta}\bar\sigma)\bar{M}_{\dot\alpha\dot\beta} \ ,
\end{aligned}
$$

where σ is an arbitrary covariantly chiral scalar superfield, $\bar{\mathcal{D}}_{\dot\alpha}\sigma = 0$. The torsion tensors of curved superspace transform as follows:

$$
\begin{aligned}
\delta_\sigma R &= 2\sigma R + \frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\bar\sigma \ , \\
\delta_\sigma G_{\alpha\dot\alpha} &= \frac{1}{2}(\sigma+\bar\sigma)G_{\alpha\dot\alpha} + \mathrm{i}\mathcal{D}_{\alpha\dot\alpha}(\sigma - \bar\sigma) \ , \\
\delta_\sigma W_{\alpha\beta\gamma} &= \frac{3}{2}\sigma W_{\alpha\beta\gamma} \ .
\end{aligned}
$$

Superconformal transformations

The vector components of ξ^A obey the master equations

$$D_{(\alpha}\xi_{\beta)\dot\beta} = 0 \quad \Longleftrightarrow \quad \bar D_{(\dot\alpha}\xi_{\dot\beta)\beta} = 0$$

which contain all information about ξ^A.
The most general conformal Killing supervector field

$$\xi_+^{\dot\alpha\alpha} = a^{\dot\alpha\alpha} + \frac{1}{2}(\sigma + \bar\sigma)\, x_+^{\dot\alpha\alpha} + \bar K^{\dot\alpha}{}_{\dot\beta}\, x_+^{\dot\beta\alpha} + x_+^{\dot\alpha\beta} K_\beta{}^\alpha + x_+^{\dot\alpha\beta} b_{\beta\dot\beta} x_+^{\dot\beta\alpha} + 4\mathrm{i}\, \bar\epsilon^{\dot\alpha}\theta^\alpha - 4x_+^{\dot\alpha\beta}\eta_\beta\theta^\alpha \ ,$$

$$\xi^\alpha = \epsilon^\alpha + \big(\bar\sigma - \frac{1}{2}\sigma\big)\theta^\alpha + \theta^\beta K_\beta{}^\alpha + +\theta^\beta b_{\beta\dot\beta} x_+^{\dot\beta\alpha} - \mathrm{i}\,\bar\eta_{\dot\beta} x_+^{\dot\beta\alpha} + 2\theta^2\eta^\alpha \ ,$$

where

$$\xi_+^a = \xi^a + \frac{\mathrm{i}}{8}\xi\sigma^a\bar\theta \ , \qquad \bar\xi^a = \xi^a \ ,$$

and $x_+^a = x^a + \mathrm{i}\theta\sigma^a\bar\theta$ is bosonic coordinate on chiral subspace of $\mathbb{M}^{4|4}$.
$\sigma = \tau - \frac{2}{3}\mathrm{i}\varphi$ describes scale (τ) and R-symmetry (φ) transformations.

Superconformal primary superfields

A tensor superfield $\mathcal{T}$ (with its indices suppressed) is said to be superconformal primary of weight (p, q) if it transforms as

$$\delta_\xi \mathcal{T} = \Big(\xi + \frac{1}{2} K^{bc}[\xi] M_{bc}\Big)\mathcal{T} + \Big(p\sigma[\xi] + q\bar{\sigma}[\xi]\Big)\mathcal{T}$$

for some parameters p and q.
Dimension of $\mathcal{T}$ is equal to $(p+q)$;
R-symmetry charge of $\mathcal{T}$ is proportional to $(p-q)$.

If $\mathcal{T}$ is superconformal primary and chiral, $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{T} = 0$, then it cannot possess dotted indices, i.e. $\bar{M}_{\dot\alpha\dot\beta}\mathcal{T} = 0$, and it must hold that $q = 0$.

In the chiral case, it suffices to say that $\mathcal{T}$ is superconformal primary of dimension p.

Superconformal action principles

- Given a superconformal primary real scalar $\mathcal{L}$ of weight (1,1),

$$\delta_\xi \mathcal{L} = \xi \mathcal{L} + \big(\sigma[\xi] + \bar{\sigma}[\xi]\big)\mathcal{L} = \partial_a(\xi^a \mathcal{L}) - D_\alpha(\xi^\alpha \mathcal{L}) - \bar{D}^{\dot\alpha}(\bar{\xi}_{\dot\alpha}\mathcal{L}) ,$$

  the functional

$$S = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar\theta \, \mathcal{L}$$

  is superconformal.

- Given a superconformal primary chiral scalar $\mathcal{L}_{\rm c}$ of dimension $+3$,

$$\bar{D}_{\dot\alpha}\mathcal{L}_{\rm c} = 0 , \qquad \delta_\xi \mathcal{L}_{\rm c} = \xi \mathcal{L}_{\rm c} + 3\sigma[\xi]\mathcal{L}_{\rm c} = \partial_a(\xi^a \mathcal{L}_{\rm c}) - D_\alpha(\xi^\alpha \mathcal{L}_{\rm c}) ,$$

  the functional

$$S_{\rm c} = \int \mathrm{d}^4x \mathrm{d}^2\theta \, \mathcal{L}_{\rm c}$$

  is superconformal.

Superconformal half-integer superspin multiplets on $\mathbb{M}^{4|4}$

Let s be a positive integer. In the superspin-$(s+\frac{1}{2})$ case, the conformal prepotential $H_{\alpha(s)\dot\alpha(s)} \equiv H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s}$ is a real superfield, which is symmetric in its undotted indices and, independently, in its dotted indices. The gauge transformation law of $H_{\alpha(s)\dot\alpha(s)}$ is

$$\delta H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = \bar D_{(\dot\alpha_1}\Lambda_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_s)} - D_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} ,$$

with unconstrained gauge parameter $\Lambda_{\alpha(s)\dot\alpha(s-1)}$.

SMK, V. Postnikov & A. Sibiryakov (1993)

The $s=1$ case corresponds to linearised conformal supergravity

S. Ferrara and B. Zumino (1978)

The superconformal transformation law of $H_{\alpha(s)\dot\alpha(s)}$ is

$$\delta_\xi H_{\alpha(s)\dot\alpha(s)} = \Big(\xi + \frac{1}{2}K^{bc}[\xi]M_{bc}\Big)H_{\alpha(s)\dot\alpha(s)} - \frac{s}{2}\big(\sigma[\xi]+\bar\sigma[\xi]\big)H_{\alpha(s)\dot\alpha(s)} .$$

It is uniquely determined if one requires both $H_{\alpha(s)\dot\alpha(s)}$ and $\Lambda_{\alpha(s)\dot\alpha(s-1)}$ to be superconformal primary

Superconformal half-integer superspin multiplets

Chiral gauge-invariant field strength

$$\mathcal{W}_{\alpha_1\dots\alpha_{2s+1}} = -\frac{1}{4}\bar{D}^2\partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{\alpha_s}{}^{\dot\beta_s}D_{\alpha_{s+1}}H_{\alpha_{s+2}\dots\alpha_{2s+1})\dot\beta_1\dots\dot\beta_s}$$

proves to be superconformal primary of dimension 3/2.

Superconformal gauge-invariant action

$$S_{s+\frac{1}{2}} = \int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{W}^{\alpha_1\dots\alpha_{2s+1}}\mathcal{W}_{\alpha_1\dots\alpha_{2s+1}} + \int \mathrm{d}^4x\mathrm{d}^2\bar\theta\,\bar{\mathcal{W}}_{\dot\alpha_1\dots\dot\alpha_{2s+1}}\bar{\mathcal{W}}^{\dot\alpha_1\dots\dot\alpha_{2s+1}}$$

is superconformal.

Special case $s=1$: Linearised conformal supergravity.
S. Ferrara and B. Zumino (1978)

Identity

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{W}^{\alpha_1\dots\alpha_{2s+1}}\mathcal{W}_{\alpha_1\dots\alpha_{2s+1}} = \int \mathrm{d}^4x\mathrm{d}^2\bar\theta\,\bar{\mathcal{W}}_{\dot\alpha_1\dots\dot\alpha_{2s+1}}\bar{\mathcal{W}}^{\dot\alpha_1\dots\dot\alpha_{2s+1}}~.$$

Superconformal half-integer superspin multiplets

Wess-Zumino gauge:

$$H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s}(\theta,\bar\theta) = \theta^\beta\bar\theta^{\dot\beta} h_{(\beta\alpha_1\dots\alpha_s)(\dot\beta\dot\alpha_1\dots\dot\alpha_s)} + \bar\theta^2\theta^\beta\psi_{(\beta\alpha_1\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} - \theta^2\bar\theta^{\dot\beta}\bar\psi_{\alpha_1\dots\alpha_s(\dot\beta\dot\alpha_1\dots\dot\alpha_s)} + \theta^2\bar\theta^2 h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} ,$$

where the bosonic fields $h_{\alpha(s+1)\dot\alpha(s+1)}$ and $h_{\alpha(s)\dot\alpha(s)}$ are real.

Residual gauge freedom: $(\mathcal{H}_0 = \theta\sigma^a\bar\theta\partial_a)$

$$\bar D_{(\dot\alpha_1}\Lambda_{\alpha(s)\dot\alpha_2\dots\dot\alpha_s)} = \mathrm{e}^{\mathrm{i}\mathcal{H}_0}\Big\{ -\frac{\mathrm{i}}{2}\zeta_{\alpha(s)\dot\alpha_1\dots\dot\alpha_s} + \mathrm{i}\bar\theta_{(\dot\alpha_1}\rho_{\alpha(s)\dot\alpha_2\dots\dot\alpha_s)} - \mathrm{i}\theta_{(\alpha_1}\bar\rho_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} + \frac{s}{s+1}\theta^\beta\bar\theta_{(\dot\alpha_1}\partial_{(\beta}{}^{\dot\gamma}\zeta_{\alpha_1\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}\dot\gamma} - \frac{1}{2}\frac{s^2}{(s+1)^2}\theta_{(\alpha_1}\bar\theta_{(\dot\alpha_1}\partial^{\gamma\dot\gamma}\zeta_{\alpha_2\dots\alpha_s)\gamma\dot\alpha_2\dots\dot\alpha_s)\dot\gamma} - 2\mathrm{i}\,\theta_{(\alpha_1}\bar\theta_{(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\alpha_s)} - \frac{s}{s+1}\theta^2\bar\theta_{(\dot\alpha_1}\partial_{(\alpha_1}{}^{\dot\gamma}\bar\rho_{\alpha_2\dots\alpha_s)\gamma\dot\alpha_2\dots\dot\alpha_s)}\Big\} ,$$

where the bosonic parameters $\zeta_{\alpha(s)\dot\alpha(s)}$ and $\zeta_{\alpha(s-1)\dot\alpha(s-1)}$ are real.

Superconformal half-integer superspin multiplets

Residual gauge transformations:

$$\begin{aligned}
\delta h_{\alpha_1\dots\alpha_{s+1}\dot\alpha_1\dots\dot\alpha_{s+1}} &= \partial_{(\alpha_1(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_{s+1})\dot\alpha_2\dots\dot\alpha_{s+1})} \ , \\
\delta h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} &= \partial_{(\alpha_1(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\dot\alpha_s)} \ , \\
\delta \psi_{\alpha_1\dots\alpha_{s+1}\dot\alpha_1\dots\dot\alpha_s} &= \partial_{(\alpha_1(\dot\alpha_1}\rho_{\alpha_2\dots\alpha_{s+1})\dot\alpha_2\dots\dot\alpha_s)} \ .
\end{aligned}$$

These transformation laws correspond to conformal higher spin fields

E. Fradkin & A. Tseytlin (1985)

E. Fradkin & V. Linetsky (1989)

Superconformal integer superspin multiplets on $\mathbb{M}^{4|4}$

In the superspin-s case, superconformal multiplet is described by unconstrained prepotential $\Psi_{\alpha(s)\dot\alpha(s-1)} \equiv \Psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}}$ and its conjugate $\bar\Psi_{\alpha(s-1)\dot\alpha(s)}$.
For $s > 1$ the gauge freedom is

$$\delta\Psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}} = D_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}} + \bar D_{(\dot\alpha_1}\zeta_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_{s-1})} ,$$

with unconstrained gauge parameters $\bar\Lambda_{\alpha(s-1)\dot\alpha(s-1)}$ and $\zeta_{\alpha(s)\dot\alpha(s-2)}$.
SMK & A. Sibiryakov (1993)

Gauge-invariant chiral field strengths (including the $s = 1$ case)

$$\mathcal{W}_{\alpha_1\dots\alpha_{2s}} = -\frac{1}{4}\bar D^2\partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{\alpha_{s-1}}{}^{\dot\beta_{s-1}}D_{\alpha_s}\Psi_{\alpha_{s+1}\dots\alpha_{2s})\dot\beta_1\dots\dot\beta_{s-1}} ,$$
$$\mathcal{Z}_{\alpha_1\dots\alpha_{2s}} = -\frac{1}{4}\bar D^2\partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{\alpha_s}{}^{\dot\beta_s}D_{\alpha_{s+1}}\bar\Psi_{\alpha_{s+2}\dots\alpha_{2s})\dot\beta_1\dots\dot\beta_s}$$

Superconformal integer superspin multiplets

Superconformal transformation law of the prepotential:

$$\delta_\xi \Psi_{\alpha(s)\dot\alpha(s-1)} = \Big\{ \xi + \frac{1}{2} K^{bc}[\xi] M_{bc} - \frac{1}{2}\big(s\sigma[\xi] + (s-1)\bar\sigma[\xi]\big) \Big\} \Psi_{\alpha(s)\dot\alpha(s-1)}$$

The chiral field strengths $\mathcal{W}_{\alpha(2s)}$ and $\mathcal{Z}_{\alpha(2s)}$ proves superconformal primaries of dimension 1 and 2, respectively.

Superconformal gauge-invariant action

$$S_s = \mathrm{i} \int \mathrm{d}^4x \mathrm{d}^2\theta \, \mathcal{W}^{\alpha_1 \dots \alpha_{2s}} \mathcal{Z}_{\alpha_1 \dots \alpha_{2s}} - \mathrm{i} \int \mathrm{d}^4x \mathrm{d}^2\bar\theta \, \bar{\mathcal{W}}_{\dot\alpha_1 \dots \dot\alpha_{2s}} \bar{\mathcal{Z}}^{\dot\alpha_1 \dots \dot\alpha_{2s}} \; .$$

Identity

$$\int \mathrm{d}^4x \mathrm{d}^2\theta \, \mathcal{W}^{\alpha_1 \dots \alpha_{2s}} \mathcal{Z}_{\alpha_1 \dots \alpha_{2s}} + \int \mathrm{d}^4x \mathrm{d}^2\bar\theta \, \bar{\mathcal{W}}_{\dot\alpha_1 \dots \dot\alpha_{2s}} \bar{\mathcal{Z}}^{\dot\alpha_1 \dots \dot\alpha_{2s}} = 0 \; .$$

Superconformal gravitino multiplet (superspin-1)

For $s = 1$ the gauge freedom is

$$\delta\Psi_\alpha = D_\alpha \bar{\Lambda} + \zeta_\alpha \ , \qquad \bar{D}_{\dot{\beta}}\zeta_\alpha = 0 \ .$$

J. Gates & W. Siegel (1980)

Chiral field strengths

$$\mathcal{W}_{\alpha\beta} = -\frac{1}{4}\bar{D}^2 D_{(\alpha}\Psi_{\beta)} \ ,$$
$$\mathcal{Z}_{\alpha\beta} = -\frac{1}{4}\bar{D}^2 \partial_{(\alpha}{}^{\dot{\beta}} D_{\beta)}\bar{\Psi}_{\dot{\beta}}$$

are obviously gauge invariant.

Superconformal higher spin multiplets in supergravity

- The linearised gauge transformations are uniquely extended to curved superspace, for instance

$$\delta H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = \bar{\mathcal{D}}_{(\dot\alpha_1}\Lambda_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_s)} - \mathcal{D}_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} ~.$$

- The chiral field strengths may be uniquely lifted to curved superspace as super-Weyl primary multiplets. However, the resulting superfields are no longer gauge invariant, if the super-Weyl tensor of background superspace is non-zero, $W_{\alpha\beta\gamma} \neq 0$.
- Starting from superspin-1, super-Weyl- and gauge-invariant actions may be obtained if the supersymmetric extension of the Bach tensor

$$\begin{aligned} B^{\alpha}{}_{\dot\alpha} &= \mathrm{i}\mathcal{D}_{\beta\dot\alpha}\mathcal{D}_\gamma W^{\alpha\beta\gamma} + (\mathcal{D}_\beta G_{\gamma\dot\alpha})W^{\alpha\beta\gamma} + G_{\beta\dot\alpha}\mathcal{D}_\gamma W^{\alpha\beta\gamma} \\ &= \mathrm{i}\mathcal{D}_{\alpha\dot\beta}\bar{\mathcal{D}}_{\dot\gamma}\bar W^{\dot\alpha\dot\beta\dot\gamma} - (\bar{\mathcal{D}}_{\dot\beta}G_{\alpha\dot\gamma})\bar W^{\dot\alpha\dot\beta\dot\gamma} - G_{\alpha\dot\beta}\bar{\mathcal{D}}_{\dot\gamma}\bar W^{\dot\alpha\dot\beta\dot\gamma} \end{aligned}$$

vanishes.

Superconformal higher spin multiplets in supergravity

Example: linearised conformal supergravity
The linearised super-Weyl tensor is

$$\mathcal{W}_{\alpha\beta\gamma} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Big\{(\mathcal{D}_{(\alpha}{}^{\dot\gamma} + \mathrm{i}G_{(\alpha}{}^{\dot\gamma})\mathcal{D}_\beta H_{\gamma)\dot\gamma}\Big\} .$$

Under the gauge transformation it varies as

$$\delta_\Lambda \mathcal{W}_{\alpha\beta\gamma} = \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}^2 - 4R)\Big[\big(\mathcal{D}^\delta W_{\delta(\alpha\beta}\big)\Lambda_{\gamma)} - \mathcal{D}_{(\alpha}\big(W_{\beta\gamma)\delta}\Lambda^\delta\big)\Big] .$$

Example: superspin-5/2

$$\mathcal{W}_{\alpha_1\dots\alpha_5} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Big\{\mathcal{D}_{(\alpha_1}{}^{\dot\beta_1}\mathcal{D}_{\alpha_2}{}^{\dot\beta_2} + 3\mathrm{i}G_{(\alpha_1}{}^{\dot\beta_1}\mathcal{D}_{\alpha_2}{}^{\dot\beta_2} - 2G_{(\alpha_1}{}^{\dot\beta_1}G_{\alpha_2}{}^{\dot\beta_2}$$
$$-\frac{1}{4}([\mathcal{D}_{(\alpha_1}, \bar{\mathcal{D}}^{\dot\beta_1}]G_{\alpha_2}{}^{\dot\beta_2}) + \frac{3}{2}\mathrm{i}(\mathcal{D}_{(\alpha_1}{}^{\dot\beta_1}G_{\alpha_2}{}^{\dot\beta_2})\Big\}\mathcal{D}_{\alpha_3}H_{\alpha_4\alpha_5)\dot\beta_1\dot\beta_2}$$

Superconformal gravitino multiplet

Gauge freedom

$$\delta\Psi_\alpha = \mathcal{D}_\alpha\bar{\Lambda} + \zeta_\alpha~, \qquad \bar{\mathcal{D}}_{\dot\beta}\zeta_\alpha = 0~,$$

Covariantly chiral field strengths

$$\begin{aligned}\mathcal{W}_{\alpha\beta} &= -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_{(\alpha}\Psi_{\beta)}~,\\ \mathcal{Z}_{\alpha\beta} &= -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Big[(\mathcal{D}_{(\alpha}{}^{\dot\alpha} + \mathrm{i}G_{(\alpha}{}^{\dot\alpha})\mathcal{D}_{\beta)}\bar{\Psi}_{\dot\alpha}\Big]\end{aligned}$$

are super-Weyl primary of dimension $+1$ and $+2$, respectively.
Super-Weyl- and gauge-invariant action

$$\begin{aligned}S &= \mathrm{i}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\mathcal{W}^{\alpha\beta}\mathcal{Z}_{\alpha\beta} - 2\mathrm{i}\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,W^{\alpha\beta\gamma}\Psi_\alpha(\mathcal{D}_{\beta\dot\beta} + \mathrm{i}G_{\beta\dot\beta})\mathcal{D}_\gamma\bar{\Psi}^{\dot\beta}\\ &+ \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,E\,(\mathcal{D}_\alpha W^{\alpha\beta\gamma})(\bar{\mathcal{D}}_{\dot\beta}\Psi_\beta)\mathcal{D}_\gamma\bar{\Psi}^{\dot\beta} ~+~ \mathrm{c.c.}\end{aligned}$$

$\mathcal{E}$ and E denote the chiral and full superspace measures, respectively.

3D $\mathcal{N} = 1$ superconformal higher spin multiplets

SMK (2016)

Superconformal higher spin prepotential: $H_{\alpha(n)} = H_{\alpha_1 \dots \alpha_n}$.
Its gauge transformation

$$\delta H_{\alpha_1 \dots \alpha_n} = \mathrm{i}^n D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_n)} , \qquad n > 0$$

$H_{\alpha(n)}$ is superconformal primary of dimension $(1 - n/2)$.

Superconformal primary and gauge-invariant field strength

$$W_{\alpha_1 \dots \alpha_n}(H) := \frac{1}{2^n} \sum_{J=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n}{2J} \Box^J \partial_{(\alpha_1}{}^{\beta_1} \dots \partial_{\alpha_{n-2J}}{}^{\beta_{n-2J}} H_{\alpha_{n-2J+1} \dots \alpha_n) \beta_1 \dots \beta_{n-2J}} - \frac{\mathrm{i}}{2} \binom{n}{2J+1} D^2 \Box^J \partial_{(\alpha_1}{}^{\beta_1} \dots \partial_{\alpha_{n-2J-1}}{}^{\beta_{n-2J-1}} H_{\alpha_{n-2J} \dots \alpha_n) \beta_1 \dots \beta_{n-2J-1}} \right\} .$$

Bianchi identity

$$D^\beta W_{\beta \alpha_1 \dots \alpha_{n-1}} = 0$$

Conformal higher spin supergravity

Alternative representation

$$W_{\alpha(n)} = \frac{(-\mathrm{i})^n}{2^n} D^{\beta_1} D_{\alpha_1} \dots D^{\beta_n} D_{\alpha_n} H_{\beta_1 \dots \beta_n} \ .$$

It is completely symmetric, $W_{\alpha_1 \dots \alpha_n} = W_{(\alpha_1 \dots \alpha_n)}$, as a consequence of

$$D^{\beta} D_{\alpha} D_{\beta} = 0 \quad \Longrightarrow \quad [D_{\alpha} D_{\beta}, D_{\gamma} D_{\delta}] = 0 \ .$$

Superconformal and gauge-invariant Chern-Simons-type action

$$S_{\rm CS}[H] = \mathrm{i}^n \int \mathrm{d}^{3|2} z \, H^{\alpha(n)} W_{\alpha(n)}(H)$$

$n = 1$ vector multiplet
$n = 3$ linearised conformal supergravity

Massive higher spin supermultiplets

SMK & M. Tsulaia (2017)

$$S^{(n/2)}_{\rm massive} = \int {\rm d}^{3|2}z \left\{ \mathcal{L}_{n/2}\big(H_{\alpha(n)}, \mathcal{X}_{\alpha(2\lfloor n/2\rfloor-2)}\big) + {\rm i}^n \lambda H^{\alpha_1\dots\alpha_n} W_{\alpha_1\dots\alpha_n}(H) \right\}$$

Here

$$S^{(n/2)}_{\rm massless} = \int {\rm d}^{3|2}z \, \mathcal{L}_{n/2}\big(H_{\alpha(n)}, \mathcal{X}_{\alpha(2\lfloor n/2\rfloor-2)}\big)$$

is gauge invariant massless action, with $\mathcal{X}_{\alpha(2\lfloor n/2\rfloor-2)}$ the compensator.

The equations of motion imply that $W_{\alpha(n)}$ is an on-shell massive superfield,

$$\Big(\frac{\rm i}{2}D^2 + m\sigma\Big) W_{\alpha(n)} = 0 ~, \qquad \sigma = (-1)^{\lfloor n/2\rfloor} \frac{\lambda}{|\lambda|} ~,$$

and hence the superhelicity of $W_{\alpha(n)}$ is $\kappa = \left(\frac{1}{2}n + \frac{1}{4}\right)\sigma$
(helicity values $\frac{n}{2}\sigma$ and $\frac{n+1}{2}\sigma$)

On-shell massive supermultiplets

For $n > 0$, a massive superfield $T_{\alpha(n)}$ is defined to be a real symmetric rank-n spinor, $T_{\alpha_1 \cdots \alpha_n} = \bar{T}_{\alpha_1 \dots \alpha_n} = T_{(\alpha_1 \cdots \alpha_n)}$, which obeys equation

$$D^\beta T_{\beta\alpha_1 \cdots \alpha_{n-1}} = 0 \quad \Longrightarrow \quad \partial^{\beta\gamma} T_{\beta\gamma\alpha_1 \dots \alpha_{n-2}} = 0 \ ,$$
$$-\frac{\mathrm{i}}{2} D^2 T_{\alpha_1 \dots \alpha_n} = m\sigma T_{\alpha_1 \dots \alpha_n} \ , \qquad \sigma = \pm 1 \ .$$

SMK, J. Novak & G. Tartaglino-Mazzucchelli (2015)