The Fermi-Pasta-Ulam recurrence and modulation instability

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OUTLINE

- Introduction: history of the problem
- Cnoidal waves
- Modulation instability
- The FPU recurrence

Introduction

The phenomenon of recurrence in nonlinear systems with many degrees of freedom was first observed in numerical experiment by Fermi, Pasta and Ulam in 1954 (more exactly the authors are Fermi – Pasta – Tsingou – Ulam; Mary Tsingou conducted numerical experiments). A non-linear term was quadratic in one test, cubic in another, and a piecewise linear approximation to a cubic in a third. The idea of Fermi was to ascertain how randomization due to the nonlinear interaction leads to the energy equipartition between large number of degrees of freedom in the mechanical chain. The length of the chain achieved $N = 64$ oscillators and long-wave initial conditions were used.

Introduction

Instead of the energy equipartition numerics showed after some definite time recurrence to the initial data accompanied by a quasi-periodic energy exchange between several initially exited modes. Since that time this problem became known as the Fermi-Pasta-Ulam (FPU) problem and was one of the most attractive subjects for numerous investigations. Later, mainly by efforts of N. Zabusky, these results were repeated by means of more powerful computers. Besides, there were observed many other peculiarities in this problem (for details see the original papers of Zabusky & Kruskal (1965), Deem & Zabusky (1966)). It was a time of forerunner of the era of integrability for nonlinear systems.

Introduction

Since the discovery of the IST for the KDV by Gardner, Greene, Kruskal and Miura (1967)

$$u_t + u_{xxx} + 6uu_x = 0,$$

and later for the nonlinear Schrodinger (NLS) equation by Zakharov and Shabat (1971),

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0,$$

many aspects of the FPU recurrence became more clear. In 1971 Zakharov and Faddeev proved that the KDV equation, which, in particular, can be obtained from the FPU system in the continuous limit for waves propagated in one direction, represents completely integrable Hamiltonian system.

Introduction

Later, in 1974 Zakharov demonstrated that the so-called nonlinear string equation (or the Boussinesq equation) which can be considered as the direct continuous limit for the FPU system also belongs to the systems integrable by the IST. According to Zakharov the long-time randomization for the FPU system can be explained by the "distance" of that system to the nearest fully integrable one. In this case its dynamics will follow in accordance with the nearest integrable system up to the moment when the deviation from the integrable trajectory can change up to the order of 1 and this time can be taken as estimate randomization time.

Introduction

We give qualitative arguments to explain the FPU analog for the NLS and its connection with the modulation instability. Analytically there are known a lot of exact solutions (K. (1977), Peregrine (1983), Akhmediev, Eleonsky & Kulagin (1985), Zakharov & Gelash (2012, 2013), etc.) describing nonlinear stage of the MI. All these solutions show the recurrence of the condensate after its interaction with solitons. After leaving solitons the condensate recovers with the same amplitude but different phase. This is the analog of the FPU recurrence for NLS.

Introduction

The FPU recurrence in NLS is very important from the practical point of view. In 1973, Hasegawa and Tappet suggested using a NLS soliton propagating in an optical fiber as an information bit. One year before Zakharov and Shabat established that the NLS solitons were shown to behave as stable particles: their forms are perturbed neither by small disturbances nor by strong interactions of the type of soliton scattering.

The current practical needs in fiber communications requires an increase in the information rates and consequently a denser packing of information. In such a case, soliton overlapping in soliton-based fiber communications becomes very important.

The NLSE has an exact solution in the form of the soliton train–the so-called cnoidal wave. This is a whole family depending on four parameters.

Introduction

The soliton solution itself belongs to this family; it can be obtained from the cnoidal wave in the limit of an infinite spatial period. Another limit of the cnoidal wave is a solution in the form of a monochromatic wave or condensate. Such a solution is known to be unstable relative to the modulation instability (MI). Less known about the MI of cnoidal waves which appears due to the soliton overlapping. For sufficiently large distance between solitons in the cnoidal wave the growth rate occurs exponentially small, but increases with the distance decrease (K. & Spector, 1999). Thus, for denser soliton packing the modulation instability can destroy information.

In this paper we show that the FPU recurrence for cnoidal wave does exist that can help to save the information, in spite of the MI.

Cnoidal waves

We start from the cnoidal wave solution in the form $\psi(t,x) = e^{i\lambda^2 t}\psi_0(x)$ for the 1D NLS,

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0.$$

where $\psi_0(x)$ is assumed to be real and obeys the Newton equation,

$$\psi_0'' = -\frac{\partial U}{\partial \psi_0}$$

in the potential $U = \frac{1}{2}\left(-\lambda^2{\psi_0}^2 + \psi_0^4\right)$. In this case the stationary NLS has one integral - energy $\varepsilon = \frac{(\psi_0')^2}{2} + U(\psi_0)$. The two well known solutions of this equation are soliton: $\psi_0 = \lambda \mathrm{sech}(\lambda x)$($\varepsilon = 0$) and condensate: $\psi_0 = \lambda/\sqrt{2}$.

Cnoidal waves

These solutions, indeed, represent two limiting solutions of the one-parameter family, which is nothing more than cnoidal wave which can be expressed in terms of elliptic functions. By introducing $I = \psi_0^2$ this equation can be rewritten first as

$$I'^2 = 4(-I^3 + \lambda^2 I^2),$$

and after shifting $I = -[\wp(x - i\omega') - \wp(a)]$, where $\wp(a) = -\lambda^2/3$, this equation transforms into the equation for the elliptic Weierstrass function:

$$(\wp')^2 + U_\wp = 0,$$

where

$$U_\wp = -4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Cnoidal waves

Here $e_{1,2,3}$ are values of $\wp$ in points $z=\omega,\omega+i\omega',i\omega'$ and equal to

$$e_1=\frac{\lambda^2}{3},\ e_{2,3}=-\frac{\lambda^2}{6}\pm\sqrt{\frac{\lambda^4}{4}+2\varepsilon}.$$

Real period 2ω is defined by oscillations between two reflection points $e_{2,3}$, $2i\omega'$ is defined by oscillations for «imaginary time» between $e_{1,2}$. As known the Weierstrass elliptic function can be represented in the form of the soliton lattice (see, Whitteker & Watson or i.e. K. & Mikhailov, 1974):

$$\wp(x+i\omega')=-\frac{\pi^2}{4\omega'^2}\left\{\sum_{n=-\infty}^{\infty}\cosh^{-2}\left[\frac{\pi\left(x-2n\omega\right)}{2\omega'}\right]+\sum_{n}{}'\sinh^{-2}\left[\frac{\pi\omega}{\omega'}n\right]-\frac{1}{3}\right\}$$

Cnoidal waves

For intensity we have

$$I = \frac{\pi^2}{4\omega'^2} \sum_{n=-\infty}^{\infty} \left\{ \cosh^{-2}\left[\frac{\pi\left(x - 2n\omega\right)}{2\omega'}\right] + \sinh^{-2}\left[\frac{\pi\omega}{\omega'}\left(n - \frac{1}{2}\right)\right]\right\}$$

I reaches its minimum at at half-distance between neighboring solitons:

$$I_{min} = \frac{\pi^2}{4\omega'^2} \sum_{n=-\infty}^{\infty} \left\{ \cosh^{-2}\left[\frac{\pi\omega}{2\omega'}\left(n - \frac{1}{2}\right)\right] + \sinh^{-2}\left[\frac{\pi\omega}{\omega'}\left(n - \frac{1}{2}\right)\right]\right\}.$$

This constant pedestal is a result of overlapping between solitons.

Cnoidal waves

At large distance between solitons their overlapping is weak and respectively $I_{\min}$ becomes exponentially small:

$$I_{\min} = \frac{4\pi^2}{\omega'^2} \exp\left(-\frac{\pi\omega}{\omega'}\right).$$

In another limit $\omega'/\omega \to \infty$, when the size of soliton in the lattice tends to infinity, overlapping between solitons becomes the main factor defining the cnoidal wave form: function $\wp(x - i\omega')$ in this case tends to the constant value $-\lambda^2/6$, corresponding to the condensate solution , plus small harmonic oscillations,

$$\wp(x - i\omega') \simeq -\lambda^2/6 + \sqrt{\varepsilon - \varepsilon_{\min}} \cos k_0 x$$

with $k_0 = \pi/\omega = \sqrt{2}\lambda$ and $e_2 \to e_3$.

Modulation instability

As well known the condensate solution is unstable with the growth rate

$$\gamma = k\sqrt{2\lambda^2 - k^2}.$$

This growth rate can be easily obtained from the so-called Bogolyubov spectrum. This spectrum follows after linearization of the defocusing NLSE,

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0,$$

on the background of the condensate $\psi = \psi_0 e^{i\lambda^2 t}$. This procedure gives the linear waves with the following dispersion relation

$$\omega = k\sqrt{2\lambda^2 + k^2},$$

Modulation instability

which at $k \to 0$ gives acoustic waves: $\omega = kc_s$ where $c_s = \sqrt{2\lambda^2}$. The MI is also known as Benjamin-Feir instability which was found first time from linear stability analysis for the periodic Stokes waves for the deep water case.
Cnoidal wave is also unstable relative to the MI (see K. & Spector, Theor. Mat. Phys. **120**, 222-236 (1999)).
To find the growth rate of the MI in this case one needs to solve the NLS equation linearized on the background of a the cnoidal wave $\psi_0(x,t)$ by setting

$$\psi(x,t) = \psi_0(x,t) + \phi$$

where ϕ is a small perturbation.
It is turned out that using the IST simplify solution of this linear problem. Following [K., Spector& Falkovich, 1984] now we show how this system arises from the auxiliary linear differential equations.

Modulation instability

The NLSE can be represented as a compatibility condition for two linear problems for the two-component vector function Ψ,

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= L\Psi = i(\lambda \sigma_3 + \hat{\psi})\Psi, \\ \frac{\partial \Psi}{\partial t} &= A\Psi = i(2\lambda^2 \sigma_3 + 2\lambda \hat{\psi} + \hat{Q})\Psi, \end{aligned}$$

where λ is spectral parameter and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\psi} = \begin{pmatrix} 0 & \psi^* \\ \psi & 0 \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} -|\psi|^2 & -i\psi_x^* \\ i\psi_x & |\psi|^2 \end{pmatrix}.$$

Modulation instability

As soon as ψ satisfies the NLSE, a compatible solution of this system exists for all ψ. We now consider two linear compatible equations for the 2×2 matrix function Φ:

$$\frac{\partial \Phi}{\partial x} = L\Phi - \Phi L, \quad \frac{\partial \Phi}{\partial t} = A\Phi - \Phi A.$$

Perturbation ϕ is provided by the relation

$$\begin{pmatrix} 0 & \phi^* \\ \phi & 0 \end{pmatrix} = [\sigma_3, \Phi].$$

which can also be verified by direct calculation. For this, the diagonal part of the compatibility Eqs and the spectral parameter λ must be excluded from this system. Simple algebra then yields linearized equations.

Modulation instability

This scheme is nothing more as the linearized version of the Zakharov-Shabat dressing procedure (suggested first time in K.& Mikhailov, 1974 and later in K., Spector & Falkovich, 1984).

To find Φ we first make the matrices $L \to L_0$ and $A \to A_0$ time independent. This is a transformation (rotation) making these matrices periodic functions in x. Therefore, a solution of this system has the form

$$\Phi_0(t, x) = \Theta(x) e^{\gamma t},$$

where $\Theta(x)$ is of the Bloch form with the quasi momentum p,

$$\Theta(x) = \theta_p e^{ipx}, \quad \theta_p(x + 2\omega) = \theta_p(x).$$

Modulation instability

Solvability condition to this system gives the dispersion relation $\gamma = \gamma(p)$. To find it one needs to solve pure algebraic equation

$$\gamma\Theta = [A_0, \Theta].$$

Omitting then all very interesting calculations we present the final answer for the maximal growth rate when the distance between solitons becomes large enough:

$$\Gamma = 8\left(\frac{\pi}{\omega'}\right)^2 \exp\left(-\frac{\pi\omega}{\omega'}\right)$$

This expression shows that Γ in this case is exponentially small but grows with the spatial period decrease.
In another limit of the condensate we arrive at the classical expression for the growth rate of the MI.

The FPU recurrence

Qu: What happens at the nonlinear stage of the MI?
We know that, according to Zakharov and Shabat, the phase space of the NLSE represents discrete number of degrees of freedom – solitons (these are the most nonlinear objects) and continuous spectrum solutions. Moreover, we know that the interaction between solitons is elastic and pairing. For scattering of two solitons it results in changes only of two soliton parameters, i.e. coordinates of center of mass and phases:

$$\Delta x_1^{(0)} = \frac{1}{2\eta_1} \log \left| \frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2} \right|^2 ,$$

$$\Delta \phi_1^{(0)} = 2 \,\mathrm{arg} \left(\frac{\lambda_1 - \lambda_2^*}{\lambda_1 - \lambda_2} \right) ,$$

where $\eta = \mathrm{Im}\,\lambda > 0$ and * means complex conjugation.

The FPU recurrence

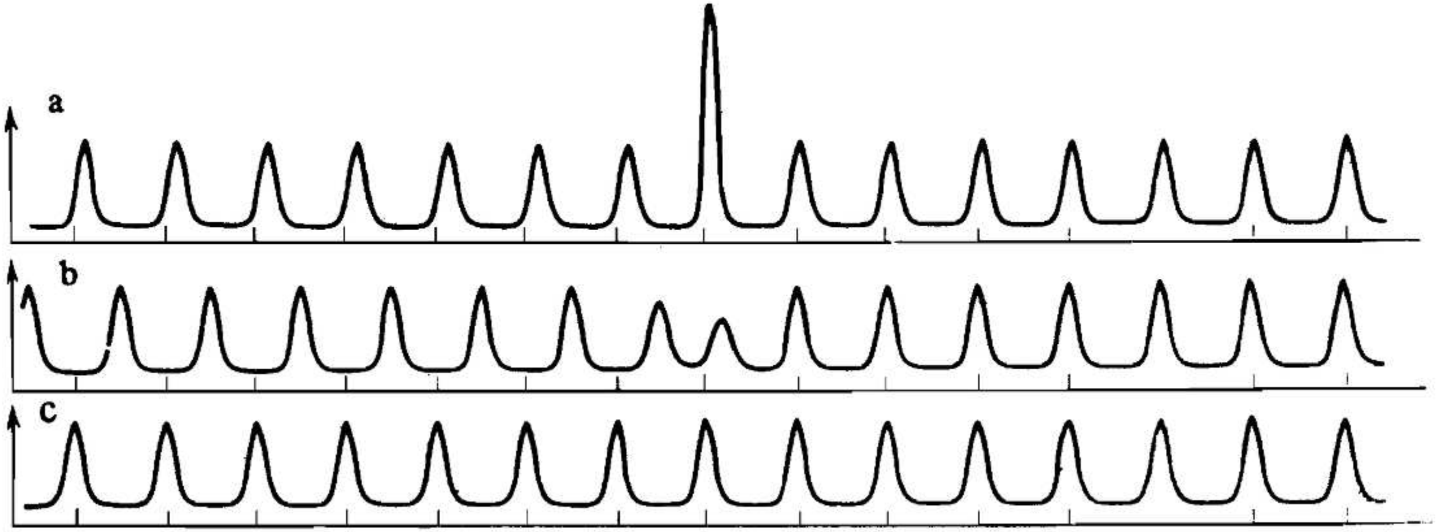
The cnoidal wave has the form of the soliton lattice. Therefore any soliton from the lattice after interaction with a soliton propagating along the cnoidal wave will undergo the same shift for its center of mass and phase. This means that after scattering of the propagating soliton with the lattice the cnoidal wave will restore its previous form, up to the definite spatial and phase shifts. Evidently, the same statement will be valid for condensate as the partial solution of the cnoidal wave. The interaction of condensate with any soliton after its propagation will restore amplitude of the condensate but its (constant) phase will be different. Scattering of a soliton with the nonsoliton part also remains the soliton form with a change of center of mass of the soliton and its phase. Thus, the cnoidal wave undergoing by the modulation instability, at the nonlinear stage, should recover its form getting some phase and spatial shifts.

The FPU recurrence

For the condensate solution the nonlinear stage of the MI leads to the recurrence of the condensate with additional (constant) phase shift.
This is the qualitative explanation of the FSU recurrence for the cnoidal wave and for the condensate, in particular.
It is necessary to underline that the same phenomenon takes place for the KDV cnoidal wave that was found by Kuznetsov & Mikhailov in 1974).

The FPU recurrence

Solitons propagating along the KDV cnoidal wave



a) Soliton in the form of point dislocation with eigen value from the first forbidden zone.
b) Soliton in the form of point dislocation with eigen value from the second forbidden zone.
c) Cnoidal wave.

Conclusion

- We have presented qualitative explanation for the FPU recurrence in the NLS equation for the cnoidal waves in the presence of perturbations, which are not assumed to be small.
- The recurrence of the condensate during nonlinear development of modulation instability represents the partial case of the recurrence for the cnoidal wave resulting in appearance of the phase shift only.
- Thus, the cnoidal waves as solitonic lattices occupy the special place in nonlinear dynamics of the system, playing the similar role like separate solitons.

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