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Colored Gravity in Three Dimensions

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Based on work with S. Gwak, E. Joung and S.-J. Rey
arXiv: 1511.05220, 1511.05975

Colored Gravity: What is it?

- Gravity: a theory that describes massless spin two field with self-interaction, interaction with matter.

- Colored Gravity: Any hypothetical theory that involves more than one massless spin two fields in its spectrum, non-trivially interacting with each other, and with matter.

Ah.. those No-Go theorems!

There are powerful no-go results, constraining the possibility of a sensible theory of multiple massless spin two particles with non-trivial interactions:

Wald '86-'87, Boulanger, Damour, Gualtieri, Henneaux '01,...

Somewhat similar to massless higher spins. First yes-go examples (with higher spins): Vasiliev at al '88-'90

Massless spin two fields: Fierz-Pauli action

Fierz-Pauli action is the linearization of Einstein-Hilbert action (around Minkowski background):

$$\mathcal{L}_{\mathcal{F}} = \frac{1}{2} h^{\mu\nu}(x) \left\{ \mathcal{F}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \mathcal{F}_{\rho}{}^{\rho} \right\}$$

"Spin two Fronsdal tensor" is the linearized Ricci tensor.

Free equation of motion is equivalent to:

$$\mathcal{F}_{\mu\nu} := \Box\, h_{\mu\nu} - 2\,\partial_{(\mu}\,\partial^{\rho}\, h_{\nu)\rho} + \frac{1}{2}\,\partial_{\mu}\partial_{\nu}\, h_{\rho}{}^{\rho} = 0$$

Gauge symmetry

$$\delta^{(0)}_{\epsilon}\, h_{\mu\nu}(x) = 2\,\partial_{(\mu}\,\varepsilon_{\nu)}(x)$$

Color decorated Gupta program

Order by order reconstruction of interacting theory of a single massless spin two particle leads to full nonlinear Einstein-Hilbert action (up to field redefinitions). (Gupta '52, Kraichnan '55, Feynman at al '63, Ogevetsky and Polubarinov '65, Deser '70)

Similar construction works for many copies of spin one fields – Yang-Mills theory.

What happens if one tries to construct a theory with multiple massless spin two fields?

Cubic vertex for massless spin two

Simplest form of the cubic vertex for massless spin two field interactions with two derivatives (Manvelyan, KM, Ruehl '10):

$$\mathcal{L}_3 = g_{IJK}\left(\tfrac{1}{2} h^{I\,\alpha\beta}\partial_\alpha\partial_\beta h^{J\,\mu\nu} h^K_{\mu\nu} + h^{I\,\alpha\mu}\partial_\alpha h^{J\,\beta\nu}\partial_\beta h^K_{\mu\nu}\right) + O(D_\mu)$$

De Donder tensor: $D_\mu = \partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h_\nu{}^\nu$

The vertex operator is symmetric (equivalent to Einstein-Hilbert vertex up to a simple field redefinition when all three fields are the same).

If this is the only non-abelian vertex in the theory, we can take the structure constants to be fully symmetric.

Global symmetry algebra

Assume we have a theory of gravity, with a global symmetry algebra (e.g., Poincaré)

$$[M_{\mu\nu}, M_{\rho\sigma}] = 4\,\eta_{[\mu[\rho}M_{\sigma]\nu]}\,, \quad [M_{\mu\nu}, P_\rho] = 2\,\eta_{\rho[\mu}P_{\nu]}$$

What are the constraints on the *isometry and color algebras*, that allow for color decorated gravitational theory?

$$[M^I_{\mu\nu}, M^J_{\rho\sigma}] = 4\,g^{IJ}{}_K\,\eta_{[\mu[\rho}M^K_{\sigma]\nu]}\,, \quad [M^I_{\mu\nu}, P^J_\rho] = 2\,g^{IJ}{}_K\,\eta_{\rho[\mu}P^K_{\nu]}$$

Let us first assume more general isometry algebra than just Poincaré algebra, and also assume that the colored gravity global symmetry algebra is a direct product of the isometry and color algebras:

$$g = g_i \otimes g_c$$

Global symmetry algebra

An element of the product algebra will be:

$$F_{X,I} = M_X \otimes T_I \,, \quad M_X \in g_i \,, \quad T_I \in g_c$$

The commutator will take a form:

$$[M_X \otimes T_I, M_Y \otimes T_J] = \frac{1}{2}[M_X, M_Y] \otimes \{T_I, T_J\} + \frac{1}{2}\{M_X, M_Y\} \otimes [T_I, T_J]$$

For Poincare isometries, we get two conditions on color algebra: Commutativity and Associativity (Cutler, Wald '87)

No-go theorem

For a collection of massless spin two fields in flat space of any dimensions, color algebra satisfying following conditions:

1. Positive definite bilinear form (unitarity),
2. Associative and commutative (necessary if the isometry algebra is Poincaré), (Cutler, Wald '87)
3. Fully symmetric structure constants (EH cubic vertex).

Statement of the theorem:

The color algebra is a direct sum of one dimensional ideals. The action for such a theory can be written as direct sum of Einstein-Hilbert actions with no cross-interaction.
(Boulanger, Damour, Gualtieri, Henneaux '01)

“Yes-go theorem”

For associative isometry algebra (e.g., Vasiliev algebra), color decoration is compatible with:

1. Positive definite bilinear form (unitarity),
2. Associative color algebra

In general, colored gravity requires higher spins (Vasiliev at al '88-'90)

Non-trivial example: extension of (A)dS(3) algebra with identity generators, corresponding to spin one symmetries (with or without higher spins). (Gwak, Joung, KM, Rey '15)

$$g_i = gl_n \oplus gl_n$$

Three dimensional model

Chern-Simons theory of three dimensional (A)dS algebra decorated with U(N) color indices:

$$g = (gl_2 \oplus gl_2) \otimes U(N)$$

Describes an interacting theory of massless spin two and spin one fields in three dimensional (A)dS space.

The action is simply Chern-Simons:

$$\mathcal{S} = \frac{\kappa}{4\pi} \int_{M_3} Tr(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A})$$

See also talk by Wenliang Li on interesting limits of this theory

Three dimensional model

The action can be rewritten in the form:

$$\mathcal{S} = \mathcal{S}_{GR} + \mathcal{S}_{CS} + \mathcal{S}_{Matter}$$

Genuine Gravity part – Einstein-Hilbert action:

$$\mathcal{S}_{GR} = \frac{\kappa N}{4\pi\ell} \int_{M_3} \epsilon_{abc}\, e^a \wedge (d\omega^{bc} + \omega^b{}_d \wedge \omega^{dc} + \frac{\sigma}{3\ell^2} e^b \wedge e^c)$$

$$= \frac{1}{16\pi G} \int d^3x \sqrt{g}(R + \frac{2\sigma}{\ell^2}), \qquad \kappa = \frac{\ell}{4NG}$$

Chern-Simons level can be small even for large (A)dS radius.

Three dimensional model

The action can be rewritten in the form:

$$\mathcal{S} = \mathcal{S}_{GR} + \mathcal{S}_{CS} + \mathcal{S}_{Matter}$$

Spin one sector – two copies of SU(N) Chern-Simons action:

$$\mathcal{S}_{CS}(A, \tilde{A}) = \frac{\kappa\sqrt{\sigma}}{2\pi} \int_{M_3} [Tr(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \\ -Tr(\tilde{\mathcal{A}} \wedge d\tilde{\mathcal{A}} + \frac{2}{3}\tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}})] \, .$$

Chern-Simons level is quantized!

$$\kappa = \frac{\ell}{4NG}$$

Three dimensional model

The action can be rewritten in the second order form:

$$\mathcal{S} = \mathcal{S}_{CS} + \frac{1}{16\pi G}\int d^3x\sqrt{g}(R - V(\chi) + \mathcal{L}_{CM}(\chi, \nabla\chi, A, \tilde{A}))$$

With potential for colored spin two matter $\chi_{\mu\nu} \in SU(N)$

$$V(\chi) = -\frac{2\sigma}{N\ell^2}Tr(I + \chi_{[\mu}{}^{\mu}\chi_{\nu]}{}^{\nu} + \chi_{[\mu}{}^{\mu}\chi_{\nu}{}^{\nu}\chi_{\lambda]}{}^{\lambda})$$

Equations of motion

$$G_{\mu\nu} - \frac{1}{2}V(\chi)g_{\mu\nu} = 0\,, \quad \frac{\partial\mathcal{L}_{CM}}{\partial\chi_{\mu\nu}} = \frac{\partial V(\chi)}{\partial\chi_{\mu\nu}}$$

Classical vacua

We look for solutions that preserve Lorenz symmetry:

$$A = 0\,, \quad \tilde{A} = 0\,, \quad \chi_{\mu\nu} = g_{\mu\nu} X \qquad X \in SU(N)$$

Equations of motion

$$G_{\mu\nu} - \frac{1}{2} V(X) g_{\mu\nu} = 0\,, \qquad \frac{\partial V(X)}{\partial X}\Big|_{Traceless} = 0$$

The VEV for colored spin two defines cosmological constant

$$V(X) = V(\chi_{\mu\nu} = g_{\mu\nu}\, X) = -\frac{2\sigma}{N\ell^2} Tr(I + 3X^2 + X^3)$$

$$\Lambda = \frac{1}{2} V(X)$$

Rainbow vacua

Equation of motion for color matrix

$$2X + X^2 = \frac{1}{N} Tr(2X + X^2) I$$

By a redefinition of variables, simplifies

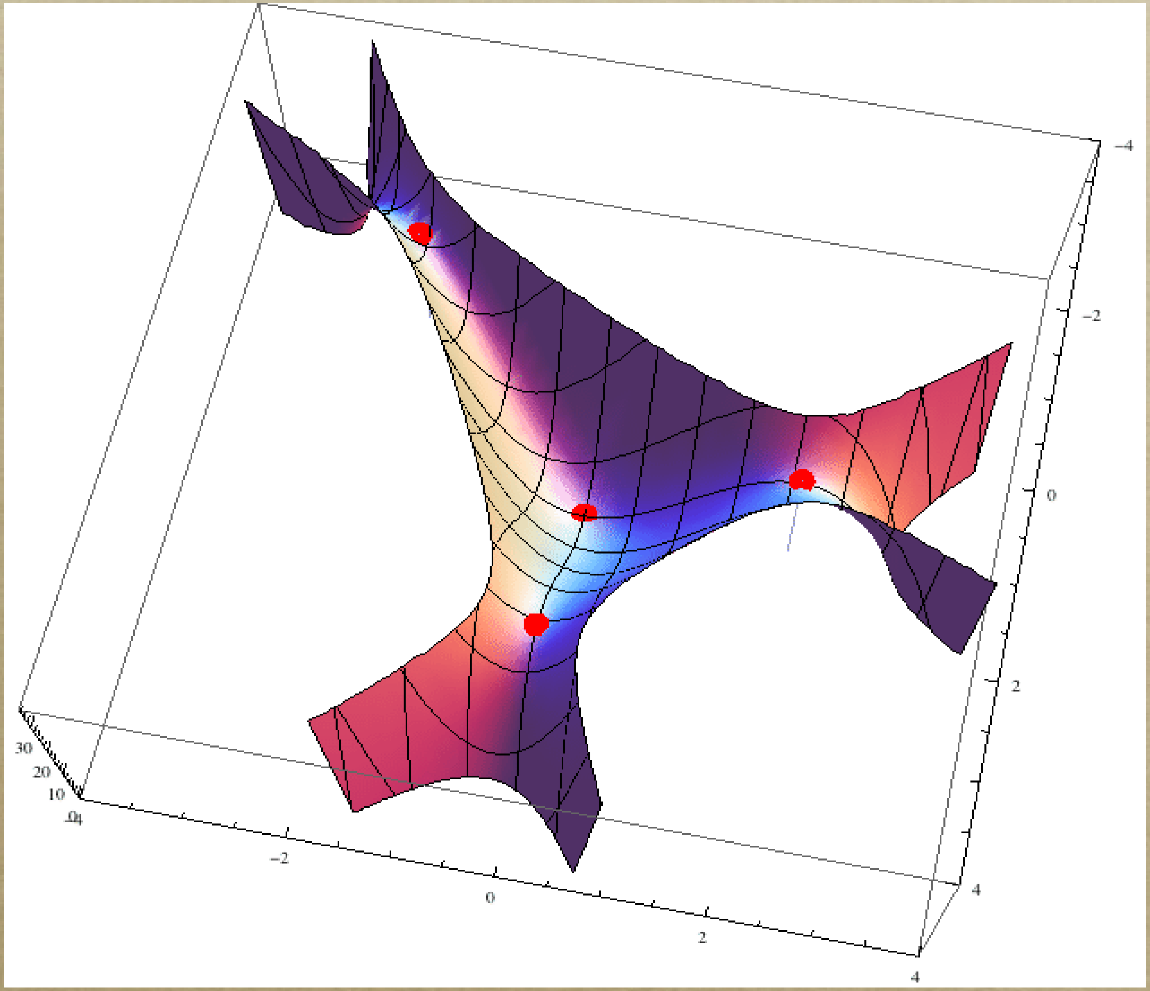
$$I + X = \frac{N}{Tr(Z)} Z \,, \quad Z^2 = I$$

All solutions, up to SU(N) rotations

$$Z_k = \begin{bmatrix} I_{(N-k)\times(N-k)} & 0 \\ 0 & -I_{k\times k} \end{bmatrix} \,, \quad k = 0, 1, \cdots, [\tfrac{N-1}{2}]$$

Example of SU(3)

Two distinct vacua up to SU(3) rotations



$$Z_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Classical vacua

The classical solutions, corresponding to

$$Z_k = \begin{bmatrix} I_{(N-k)\times(N-k)} & 0 \\ 0 & -I_{k\times k} \end{bmatrix}, \quad k = 0, 1, \cdots, [\tfrac{N-1}{2}]$$

break the color symmetry algebra

$$su(N) \to su(N-k) \oplus su(k) \oplus u(1)$$

What happens to the fields, corresponding to broken sector?

Massless spin two Goldstone fields combine with massless spin one Chern-Simons fields to form a long representation – partially massless spin two

Classical vacua

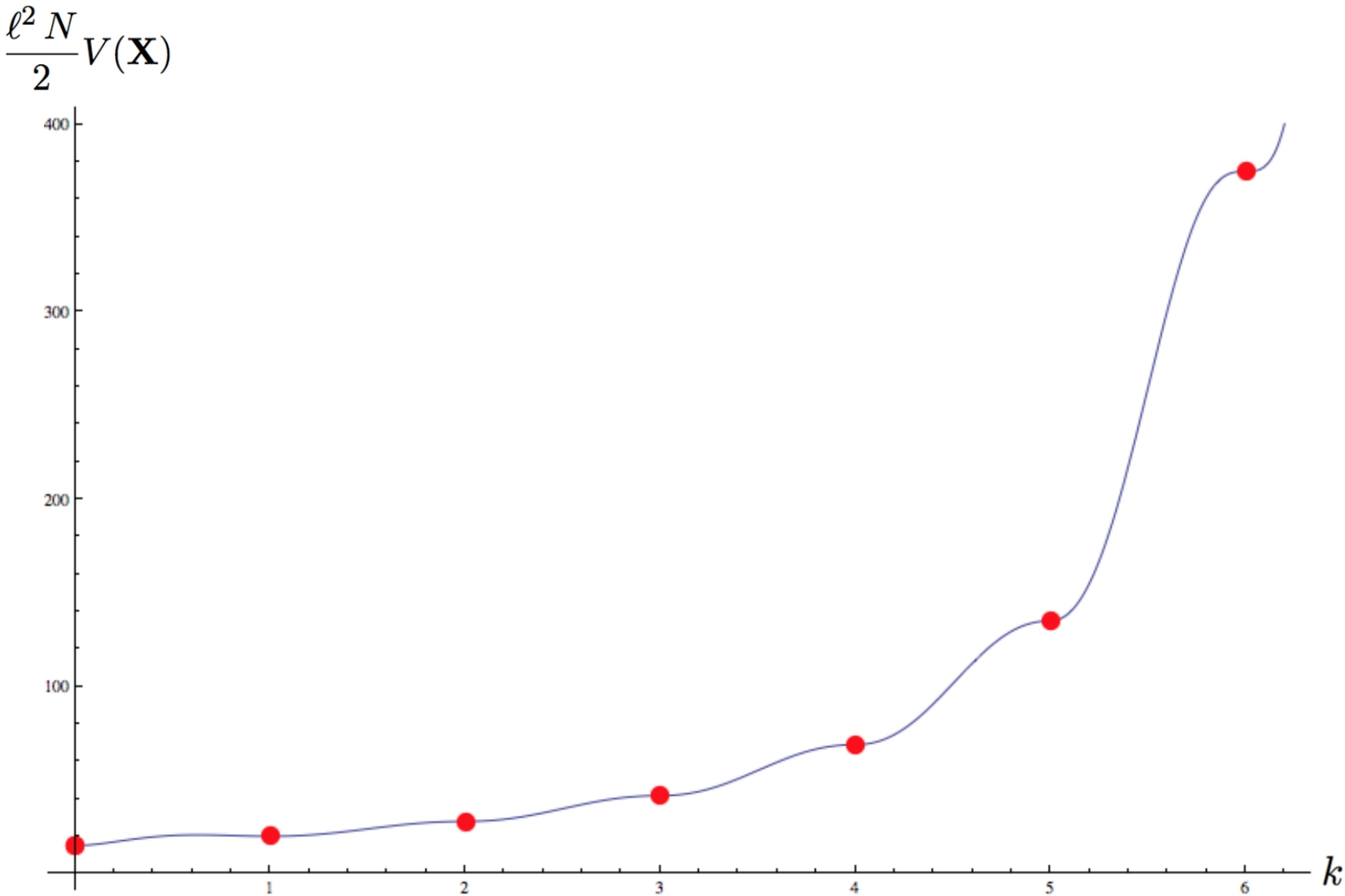
Cosmological constant in the k-th vacuum

$$\Lambda_k = -\frac{\sigma}{\ell^2}\left(\frac{N}{N-2k}\right)^2$$

Gravitational part of the action

$$S_{GR} = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2\sigma}{\ell_k^2}\right), \qquad \ell_k = \frac{N-2k}{N}\ell$$

In de Sitter case, vacua with higher value of cosmological constant are unstable and will decay to lower value. In 4d, somewhat similar scenario is phenomenologically relevant in Cosmology. But... colored gravity in 4d requires HS!



$\frac{\ell^2 N}{2} V(\mathbf{X})$
400
300
200
100
1
2
3
4
5
6
k

Thank you

Interactions: Noether Procedure
(Gupta program for gravity)

Expand the action and the gauge transformations in power of fields:

$$S\,[\varphi] = S_2\,[\varphi] + g\,S_3\,[\varphi] + g^2\,S_4\,[\varphi] \ldots$$

$$\delta\,\varphi = \delta^0\,\varphi + g\,\delta^1\,\varphi + g^2\,\delta^2\,\varphi \ldots$$

Gauge invariance of the nonlinear action $\delta\,S[\varphi] = 0$

Implies for lowest orders:

$\delta^0\,S_2\,[\varphi] = 0$ free theory

$g(\,\delta^0\,S_3\,[\varphi] + \delta^1\,S_2\,[\varphi]\,) = 0$ first order deformation

$$g^2\,(\,\delta^2\,S_2\,[\varphi] + \delta^1\,S_3\,[\varphi] + \delta^0\,S_4\,[\varphi]\,) = 0$$

$\ldots$

Noether Procedure: Cubic order analysis

Gauge invariance at the first nontrivial order:

$$\delta^0\, S_3\,[\varphi] \;+ \delta^1\, S_2\,[\varphi] \;=\; 0$$

≈ 0

Variation of the free action is always proportional to free equations of motion.

For on-free-mass-shell fields, the cubic action satisfies:

$$\delta^0\, S_3\,[\varphi] \;\approx\; 0$$

On the other hand, for Killing parameters: $\delta^0_K \varphi \;=\; 0$

$$\delta^1_K\, S_2\,[\varphi] \;=\; 0$$

Free fields carry a representation of the global symmetry of the theory. This condition strongly constrains global symmetry algebras (admissibility condition).

Konstein, Vasiliev '89; Joung, KM '14

Quartic analysis

Equation for the quartic vertex:

$$g^2 \left(\delta^2 S_2 [\varphi] + \delta^1 S_3 [\varphi] + \delta^0 S_4 [\varphi] \right) = 0$$

For Killing parameters and on-free-shell fields:

$$\delta^1_K S_3 [\varphi] \approx 0$$

Fixes all the relative couplings between the cubic vertices before even knowing anything about quartic vertex. Analogous equation in 4d light-cone theory. (Metsaev '91)
Moreover, the first order deformation closes on Killing parameters (carries a representation of the global symmetry algebra). (Joung, KM '14)

$$[\delta^1_{\epsilon_K}, \delta^1_{\eta_K}] = \delta^1_{[\epsilon_K, \eta_K]} + (trivial \quad terms)$$

Quartic Noether equations can be always solved relaxing locality.
(Barnich, Henneaux '93)