U(N) D=3 Matter Coupled to Chern-Simons Fields. Spontaneous Breaking of Scale Invariance, AdS/CFT and Fermion-Boson Dual Mapping

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This talk: mainly on the following papers

M. M. and Jean Zinn-Justin
JHEP 1501 (2015) 054

William A. Bardeen and M. M.
JHEP 1406 (2014) 113

+ some history and some recent work in progress

Plan of this talk

(1) motivation

AdS/CFT and spontaneous breaking of scale invariance

Past motivation - smallness of physical masses

(2) Old Stuff

$$\int \mathrm{d}^3x \, \left[-\frac{1}{2}\vec{\phi}\cdot\partial^2\vec{\phi} + \frac{\mu^2}{2}(\vec{\phi})^2 + \frac{\lambda}{4N}(\vec{\phi})^4 + \frac{\eta}{6N^2}(\vec{\phi})^6 \right]$$

and

$$\mathcal{S} = \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \left[\tfrac{1}{2}\bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi + NU(\Phi^2/N) \right]$$

with

$$U(\Phi^2/N) = (\mu/N)\Phi^2 + \tfrac{1}{2}(u/N^2)\Phi^4$$

(3) Chern-Simons gauge field coupled to a U(N) scalar - light cone gauge

$$\mathcal{S}_{\mathrm{CS}}(\mathbf{A}) = -\frac{i\kappa}{4\pi}\epsilon_{\mu\nu\rho}\int \mathrm{d}^3x\, Tr\left[\mathbf{A}_\mu(x)\partial_\nu\mathbf{A}_\rho(x) + \frac{2}{3}\mathbf{A}_\mu(x)\mathbf{A}_\nu(x)\mathbf{A}_\rho(x)\right]$$

$$\mathcal{S}_{\mathrm{Scalar}} = \int \mathrm{d}^3x \left[(\mathbf{D}_\mu\phi(x))^\dagger \cdot \mathbf{D}_\mu\phi(x) + NV\left(\phi(x)^\dagger \cdot \phi(x)/N\right)\right].$$

and

(4) $$\mathcal{S}(\psi,\bar{\psi},\mathbf{A}) = \mathcal{S}_{\mathrm{CS}}(\mathbf{A}) + \mathcal{S}_{\mathrm{F}}(\psi,\bar{\psi},\mathbf{A})$$

with $$\mathcal{S}_{\mathrm{F}}(\psi,\bar{\psi},\mathbf{A}) = -\int \mathrm{d}^3x\,\bar{\psi}(x)(\not{\mathbf{D}} + M_0)\psi(x)$$

In the $\mathbf{A}_3 = 0$ gauge,

(5) **On the Fermion-Boson mapping in 3D**

Will be shown that the conditions for spontaneous breaking of scale invariance in the boson and fermion theories are dual copies of each other.

Klebanov Polyakov conjecture (hep-th/0210114):

Minimal bosonic higher-spin gauge theory with even spins in $AdS_4 ---->$ Standard AdS/CFT of this action using dimension Δ_+ gives the correlation functions of the singlet currents in the large N vector $\lambda(\vec{\phi})^4$ at its **IR critical point**.

Also: *Sundborg, Sezgin, Sundell, Gubser, Mitra, Sagnotti, Vasiliev, Witten, Leigh, Petkou, Giombi, Yin*
. . . . . and many others (2002-2007)

and more recent *Aharony et al., Maldacena et al. , Wadia et al. , Giombi et al, Jevicki et al*
. . . . . and many others(2011-2015)

O(N) vector theories in d=3 SUSY and non-SUSY with Chern-Simons interaction

Two, very well understood, mechanisms for breaking scale invariance

(a) Explicit breaking of scale invariance, which is expressed at the quantum level by the anomaly in the trace of the energy momentum tensor, as the result of radiative corrections.

(b) Spontaneous breaking of scale invariance (e.g. Nambu–Jona-Lasinio as the relativistic version of the BCS theory)

In conventional quantum field theories the two mechanisms occur simultaneously, no massless Nambu Goldston boson (Dilaton).

Supersymmetric models in the large N limit

$O(N)$ invariant supersymmetric action (d=3):

$$\mathcal{S} = \int \mathrm{d}^3x \, \mathrm{d}^2\theta \ \left[\tfrac{1}{2}\bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi + NU(\Phi^2/N)\right]$$

$O(N)$ vector: $\Phi(\theta, x) = \varphi + \bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta F$

Components - for a generic super-potential:

$$\mathcal{S} = \int \mathrm{d}^3x \ \tfrac{1}{2}[-\bar{\psi}\partial\!\!\!/\psi + (\partial_\mu\varphi)^2 - (\bar{\psi}\cdot\psi)U'(\varphi^2/N)$$

$$- 2(\bar{\psi}\cdot\varphi)(\varphi\cdot\psi)U''(\varphi^2/N)/N + \varphi^2 U'^2(\varphi^2/N)]$$

The following are several phenomena that take place at $N \to \infty$:

(1) A supersymmetric ground state with $m_\psi = m_\phi \neq 0$ exists even in a renormalized **scale invariant** theory.

(2) At a certain strength of the attractive force between $O(N)$ bosons and fermions, **massless** $O(N)$ singlets bound states are created.

(3) At the, above mentioned, critical value of the coupling constant, though $m_\psi = m_\phi \neq 0$ there is no explicit breaking of scale invariance $< \partial^\mu S_\mu > \sim < \tilde{T}^\nu_{\ \nu} > = 0$.

(4) The massless fermionic and bosonic $O(N)$ singlet bound states mentioned in (2) are the Goldston-bosons and fermion (Dilaton and Dilatino) of the spontaneously broken scale invariant theory.

(5) Item number **(4)** is related to the **double scaling limit** in O(N) matrix and vector theories and the stringy nature of quantum field theory in this limit.

(6) Will discuss **finite temperature** effects on **(1)-(5)** and an unusual phase transitions in the supersymmetric model in d=3.

Large N methods for supersymmetric actions

Introduce two new superfields:

$$L(\theta, x) = M + \bar{\theta}\ell + \tfrac{1}{2}\bar{\theta}\theta\lambda$$
$$R(\theta, x) = \rho + \bar{\theta}\sigma + \tfrac{1}{2}\bar{\theta}\theta s.$$

Add an extra term to the action:

$$\mathcal{S}(\Phi, L, R) = \int \mathrm{d}^3x\, \mathrm{d}^2\theta\, \{\tfrac{1}{2}\bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi + NU(R) + L(\theta)[\Phi^2(\theta) - NR(\theta)]\}$$

Integrate out $N-1$ components of Φ, $(\Phi_1 \equiv \phi)$.

$$\mathcal{Z} = \int [\mathrm{d}\phi][\mathrm{d}R][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}_N(\phi,R,L)}$$

$$\begin{aligned}\mathcal{S}_N = \int \mathrm{d}^3x\, \mathrm{d}^2\theta\ [\tfrac{1}{2}\bar{\mathrm{D}}\phi\mathrm{D}\phi + NU(R) \\ + L(\phi^2 - NR)] \\ + \tfrac{1}{2}(N-1)\mathrm{Str}\ \ln[-\bar{\mathrm{D}}\mathrm{D} + 2L].\end{aligned}$$

Note: action $\sim N$ and thus three saddle point equations (in terms of the **superfields** ϕ, R, L):

Action density: $\mathcal{E} = \mathcal{S}_N/\text{volume}$:

$$\mathcal{E}/N = \tfrac{1}{2} M^2 \varphi^2/N + \frac{1}{24\pi}(m - |M|)^2 (m + 2|M|)$$

m is the boson mass, M is the fermion mass.

$\mathcal{E}$ is positive for all saddle points and has an absolute minimum at $m_\varphi \equiv m = |M| = m_\psi$ (a supersymmetric ground state).

Φ^4 super-potential in $d = 3$: phase structure

$$U(\Phi^2/N) = (\mu/N)\Phi^2 + \tfrac{1}{2}(u/N^2)\Phi^4$$

Gap equations (saddle point equations) reduce to

$$M = \mu - \mu_c + u\frac{\varphi^2}{N} - \frac{u}{4\pi}|M| \quad , \qquad M\varphi = 0$$

Note the special case:
$\mu - \mu_c \equiv \mu_R = 0$ in the $O(N)$ symmetric phase ($\varphi = 0$).

The gap equation is:

$$M = -\frac{u}{4\pi}|M|$$

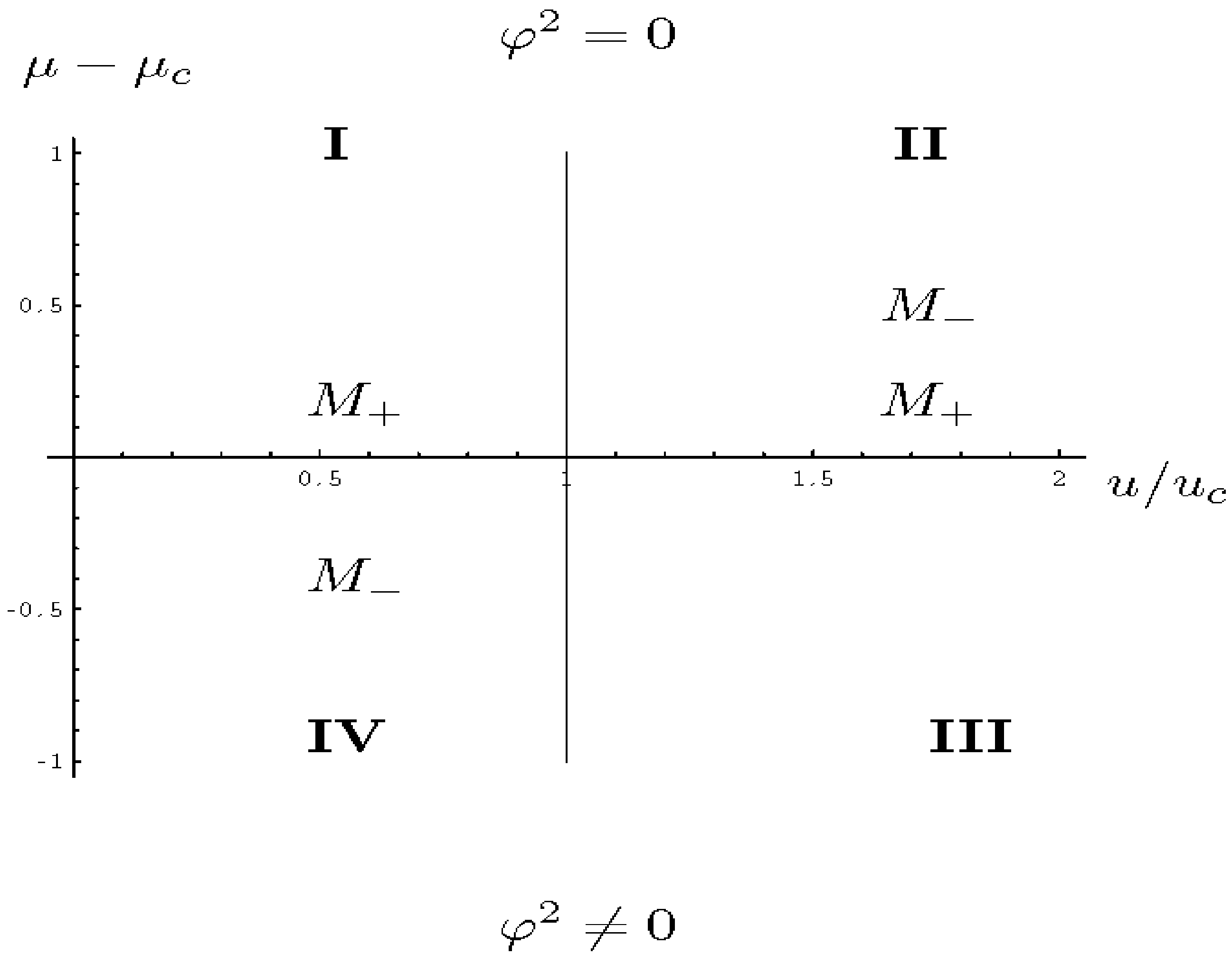


Fig. 1 Summary of the phases of the model in the $\{\mu - \mu_c, u\}$ plane. Here $m_\varphi = m_\psi = |M_\pm| = (\mu - \mu_c)/(u/u_c \pm 1)$. The lines $u = u_c$ and $\mu - \mu_c = 0$ are lines of first and second order phase transitions.

The action density $\mathcal{E}(m,\varphi)$*:*

$$\frac{1}{N}\mathcal{E}(m,\varphi) = \quad \tfrac{1}{2}M^2(m,\varphi)\frac{\varphi^2}{N}$$
$$+\ \frac{1}{24\pi}\left[m - |M(m,\varphi)|\right]^2 \times \left(m_\varphi + 2\,|M(m,\varphi)|\right)$$

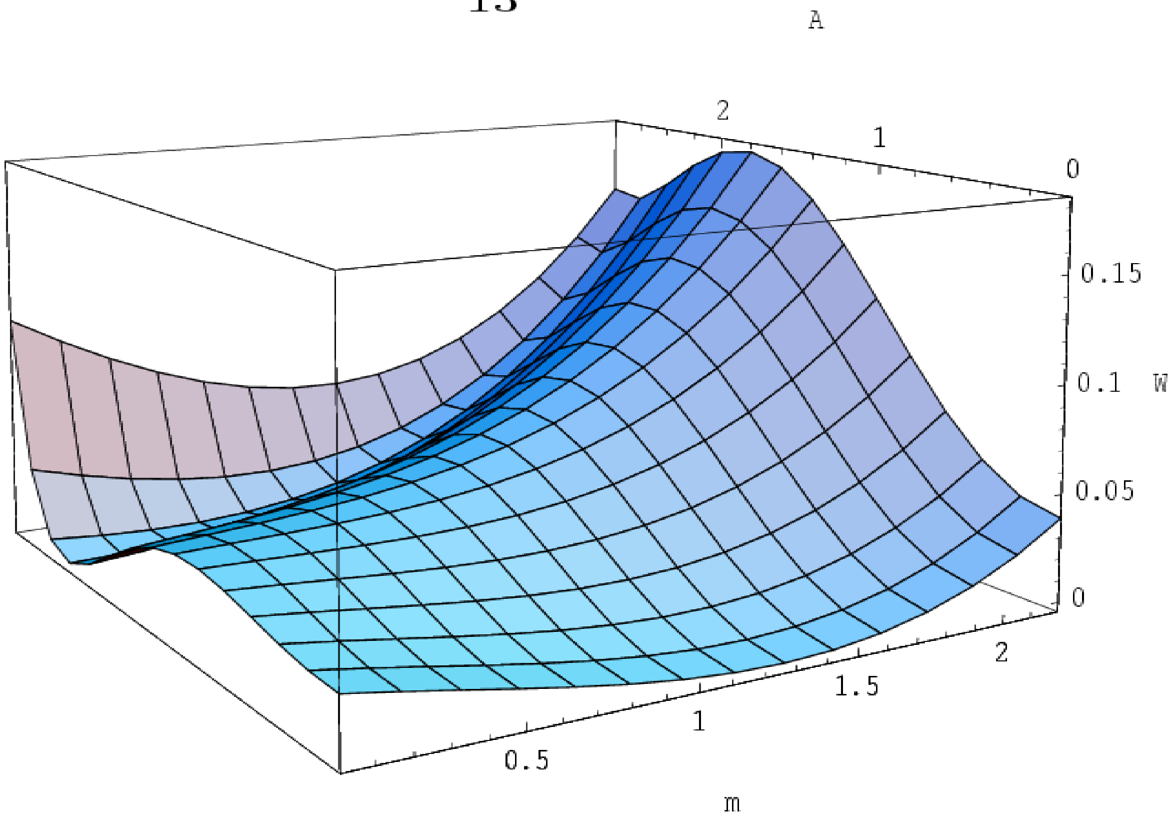


Fig. 2 **Region IV :** The action density $\frac{1}{N}\mathcal{E}(m,\varphi)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Here $\mu - \mu_c = -1$, $u/u_c = 0.2$. As seen here there are two distinct degenerate phases. One is an ordered phase ($\varphi \neq 0$) with a massless boson and fermion, the other is a symmetric phase ($\varphi = 0$) with a massive ($m = |M_-|$) boson and fermion.

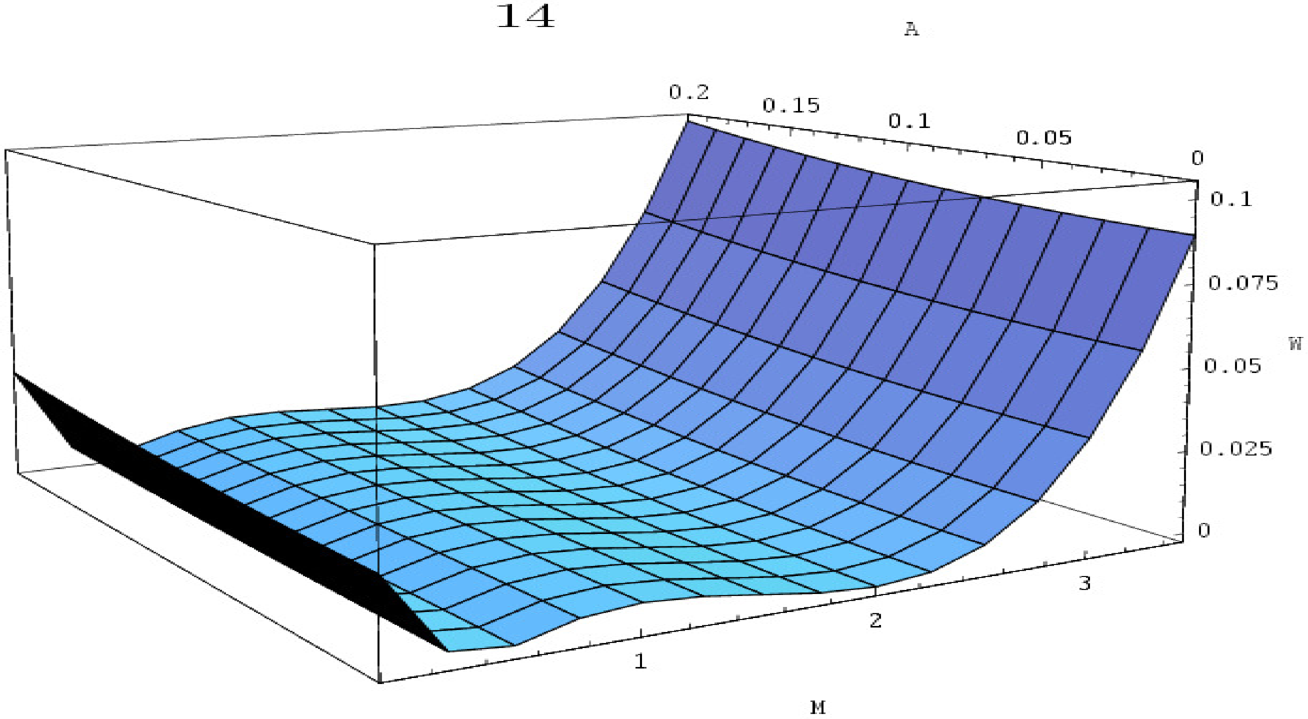


Fig. 3 **Region II:** The action density $\mathcal{E}(m, \varphi)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Two degenerate, $O(N)$ symmetric phases exist with massive bosons (and massive fermions). Here $\mu - \mu_c = 1$ (sets the mass scale) $u/u_c = 1.5$.

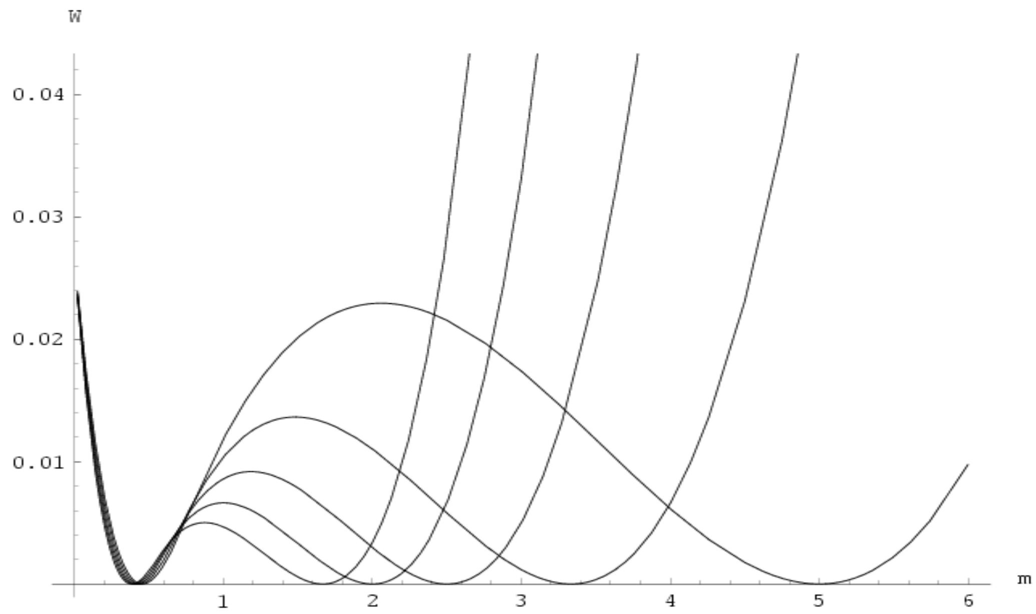


Fig. 4 The action density $\frac{1}{N}\mathcal{E}(m,\varphi = 0)$ in region **II** of Fig. 1. Here $\mu - \mu_c = 1$ (sets the mass scale) and u/u_c= varies between 1.6 and 1.2. There are two massive, degenerate $O(N)$ symmetric SUSY vacua with $\varphi = 0$ and possible large hierarchy of masses $|M_-/M_+| >> 1$ (as $u \rightarrow u_c$) with $m_\psi = m_\varphi = M_+ = (\mu - \mu_c)/(u/u_c + 1)$ and $m_\psi = m_\varphi = -M_- = (\mu - \mu_c)/(u/u_c - 1)$.

More on $u = u_c = 4\pi$:

For example in $g_0(\vec{\phi}^2)^3_{d=3}$ at $N \to \infty$ limit.

Leave cutoff Λ finite and thus ("flat potential"):

$$m^2_{phys} = \Lambda^2 F[g] = \Lambda^2 A(g_0 - g_c)$$

when $\qquad g_0(\frac{\Lambda}{\mu}) \quad \to \quad g_c \qquad$ as $\qquad \frac{\Lambda}{\mu} >> 1$

find:

$m^2_{phys} = A' \mu^2$ by "dimensional transmutation"

and define the $\beta(g_0)$ function from:

$$\frac{\partial m^2_{phys}}{\partial \Lambda} = 0$$

Still

$$< \partial_\nu S^\nu > = < T^\mu_\mu > = 0$$

there is no scale anomaly and scale invariance is spontaneously broken.

Special situation: $u = u_c = 4\pi$

$\langle LL \rangle$ propagator, and massless fermion and boson $O(N)$ singlet bound states

The $\langle LL \rangle$ action

$$-\frac{N}{2u}\int \mathrm{d}^3x\,\mathrm{d}^2\theta (L-\mu)^2 \\ +\tfrac{1}{2}(N-1)\mathrm{Str}\,\ln\left(-\bar{\mathrm{D}}\mathrm{D}+2L\right)$$

$$\Delta_L^{-1} = -\frac{N}{4\pi|M|}\{1 \\ +[M+|M|(u_c/u)]\delta^2(\theta'-\theta)\}\,\mathrm{e}^{i\bar{\theta}\not{p}\theta'}$$

Corresponds to a bound state super-particle of mass $2M(1-u_c/u)$. At the special point $u=u_c$ the mass vanishes.

Namely, massless boson and fermion, O(N) singlets associated with the spontaneous breakdown of scale invariance. Dilaton and Dilatino masses

$$m_{D_\psi} = m_{D_\phi} = 2M(1-u/u_c) \to 0$$

as $u \to u_c$

E.g. The $\psi \cdot \varphi$ scattering amplitude

$T_{\psi\cdot\varphi,\psi\cdot\varphi}(p^2)$, in the limit $p^2 \to 0$ satisfies:

$$
\begin{aligned}
T_{\psi\varphi,\psi\varphi}(p^2) &\sim \frac{-i2u}{N}\left[1 + \frac{u}{4\pi}\frac{m_\psi}{|m_\psi|} - \frac{u}{8\pi}\frac{\not{p}}{|m_\psi|}\right]^{-1} \\
&\to i\frac{16\pi}{N}\frac{|m_\psi|}{\not{p}}
\end{aligned}
$$

Namely, a masless $O(N)$ singlet, fermion-boson bound state Dilatino for $m_\psi < 0$ and $u \to u_c$
If we slightly deviate from the critical coupling u_c, dilatino acquires a mass given by

$$m_{D_\psi} = 2\left(1 - \tfrac{u_c}{u}\right)|m_\psi|$$

Similarly, in the boson-boson scattering amplitude $T_{\varphi\cdot\varphi,\varphi\cdot\varphi}$ or fermion-fermion $T_{\psi\cdot\psi,\psi\cdot\psi}$ or fermion-fermion to boson-boson scattering amplitude $T_{\psi\cdot\psi,\varphi\cdot\varphi}$ one finds the Dilaton pole at

$$m^2_{D_\varphi} = 4\left(1 - \tfrac{u_c}{u}\right)^2 m^2_\varphi$$

A supersymmetric non-linear σ-model at large N

Consider the supersymmetric n.l. σ-model in d dimensions, $2 \le d \le 3$.

$$\mathcal{S}(\Phi) = \frac{1}{2\kappa} \int \mathrm{d}^d x \, \mathrm{d}^2\theta \, \bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi$$

Dimension $d = 3$.

$$\begin{aligned} \mathcal{E}/N = {} & \frac{1}{2N\kappa} M^2 \varphi^2 \\ & + \frac{1}{24\pi} (m - |M|)^2 (m + 2|M|) \end{aligned}$$

Dimension $d = 2$.

$$\frac{\mathcal{E}}{N} = \frac{1}{8\pi} \left[m^2 - M^2 + 2M^2 \ln(M/m) \right] .$$

Contin. *Dimension* $d = 2$.

$$M = m = \Lambda \, \mathrm{e}^{-2\pi/\kappa} \, .$$

Energy momentum tensor for the SUSY Lagrangian in 3D

$$L = \tfrac{1}{2}[\nabla_\alpha\varphi\nabla^\alpha\varphi - \mu_0^2\varphi^2] + \tfrac{1}{2}\overline{\psi}(i\gamma^\alpha\nabla_\alpha - \mu_0)\psi$$
$$- (u/N)\mu_0(\varphi^2)^2 - \tfrac{(u/N)^2}{2}(\varphi^2)^3$$
$$- \tfrac{(u/N)}{2}\varphi^2(\overline{\psi}\psi) - \xi R(x)\varphi^2$$

For scalars:

$\nabla_\alpha = V_\alpha{}^\mu \frac{\partial}{\partial x^\mu}$ where $V_\alpha{}^\mu$ is a tetrad,

For fermions:

$$\nabla_\alpha = V_\alpha{}^\mu \frac{\partial}{\partial x^\mu} + \tfrac{1}{2}\sigma^{\beta\gamma}V_\beta{}^\nu V_\alpha{}^\mu V_{\gamma\nu;\mu}$$

The covariant derivative is $V_{\gamma\nu;\mu} = \frac{\partial V_{\gamma\nu}}{\partial x^\mu} - \Gamma^\lambda_{\nu\mu}V_{\gamma\nu}$ and $\sigma^{\alpha\beta} = \frac{1}{4}\left[\gamma^\alpha, \gamma^\beta\right]$ are the generators of the Lorentz group representation for spin $\frac{1}{2}$.

The action is given in tetrad formalism in $d = 3$:

$$\delta S_{matter} = \int d^3x \sqrt{-g} U^{\alpha}{}_{\mu} \delta V_{\alpha}{}^{\mu}$$

$$T_{\mu\nu} = V_{\alpha\mu} U^{\alpha}{}_{\nu}$$

where $\qquad U^{\alpha}{}_{\mu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta V_{\alpha}{}^{\mu}}$

and:

$$T_{\mu\nu}(x) = \frac{V_{\alpha\mu}(x)}{\det[V(x)]} \frac{\delta S_{matter}}{\delta V_{\alpha}{}^{\nu}}$$

Finally, in a covariant, symmetrized form in a general non-flat background, the energy momentum tensor is given by:

$$\begin{aligned}
&T^{SUSY}_{\mu\nu} = \\
&\tfrac{1}{8}\overline{\psi} i \left(\gamma_{\mu} \overrightarrow{\nabla}_{\nu} + \gamma_{\nu} \overrightarrow{\nabla}_{\mu} \right) \psi - \tfrac{1}{8}\overline{\psi} i \left(\gamma_{\mu} \overleftarrow{\nabla}_{\nu} + \gamma_{\nu} \overleftarrow{\nabla}_{\mu} \right) \psi \\
&+ \nabla_{\mu}\varphi \nabla_{\nu}\varphi - \tfrac{1}{2} g_{\mu\nu} \left(\overline{\psi} i \not{\nabla} \psi - \mu_0 \overline{\psi}\psi - (\tfrac{u}{N}) \varphi^2 \left(\overline{\psi}\psi \right) \right) \\
&- \tfrac{1}{2} g_{\mu\nu} \left(\nabla_{\rho}\varphi \nabla^{\rho}\varphi - \mu_0^2 \varphi^2 - 2(\tfrac{u}{N}) \mu_0 \left(\varphi^2 \right)^2 - (\tfrac{u}{N})^2 \left(\varphi^2 \right)^3 \right) \\
&+ \xi \left(\tfrac{g_{\mu\nu}}{2} R - R_{\mu\nu} \right) \varphi^2 + \xi \left[g_{\mu\nu} \nabla_{\alpha} \nabla^{\alpha} - \nabla_{\mu} \nabla_{\nu} \right] \varphi^2
\end{aligned}$$

The SUSY energy-momentum tensor in 3D ($\xi = \frac{1}{8}$ in 3D) reduces in the case of flat space to:

$$
\begin{aligned}
T_{\mu\nu} &= \partial_\mu\varphi\partial_\nu\varphi + \tfrac{i}{4}(\overline{\psi}\gamma_\mu\partial_\nu\psi + \overline{\psi}\gamma_\nu\partial_\mu\psi) \\
&- \eta_{\mu\nu}[\tfrac{1}{2}\partial_\alpha\varphi(x)\partial^\alpha\varphi(x) - \tfrac{\mu_0^2}{2}\varphi^2 \\
&\qquad - (u/N)\mu_0\left(\varphi^2\right)^2 - \tfrac{(u/N)^2}{2}\left(\varphi^2\right)^3] \\
&- \eta_{\mu\nu}\left(\tfrac{1}{2}\overline{\psi}i\partial\!\!\!/\psi - \tfrac{\mu_0}{2}\overline{\psi}\psi - \tfrac{(u/N)}{2}\varphi^2\left(\overline{\psi}\psi\right)\right) \\
&- \tfrac{1}{8}\left(\partial^2_{\mu\nu}\varphi^2 - \eta_{\mu\nu}\partial^2\varphi^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle p_2^a | T^{\mu\nu} | p_1^a \rangle = p_1^\mu p_2^\nu + p_2^\mu p_1^\nu \\
- \eta^{\mu\nu} p_1 p_2 - \eta^{\mu\nu} m^2 \\
+ \tfrac{1}{4} \left(q^\mu q^\nu - \eta^{\mu\nu} q^2 \right) \times \\
\times \left[1 - 8 \int_0^1 dx x (1-x) \left[1 + \tfrac{x(1-x) q^2}{m^2} \right]^{-\frac{1}{2}} \right] \times \\
\times \left[1 - \int_0^1 dx \left[1 + \tfrac{x(1-x) q^2}{m^2} \right]^{-\frac{1}{2}} \right]^{-1}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\langle p_2^a | T^{\mu\nu} | p_1^a \rangle = p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \eta^{\mu\nu} p_1 p_2 - \eta^{\mu\nu} m^2 \\
+ \left(q^\mu q^\nu - \eta^{\mu\nu} q^2 \right) \left(\tfrac{1}{4} - \tfrac{m^2}{q^2} \right)
\end{aligned}
$$

The trace of the energy-momentum tensor:

$$
\begin{aligned}
T^\mu_\mu &= 2p_1p_2 - 3p_1p_2 - 3m^2 \\
&+ \left(q^2 - 3q^2\right)\left(\tfrac{1}{4} - \tfrac{m^2}{q^2}\right) \\
&= -p_1p_2 - m^2 - \tfrac{q^2}{2} \\
&= -\tfrac{1}{2}\left((p_1 - p_2)^2 + 2p_1p_2\right) - m^2 = 0
\end{aligned}
$$

Used (here, Euclidean space) $p_1^2 = p_2^2 = -m^2$.

Energy momentum tensor in one particle fermionic state

$$
\begin{aligned}
&\int d^3x e^{iqx} \left\langle p^a_{2_\psi} \left| T_{\mu\nu}(x) \right| p^a_{1_\psi} \right\rangle = \\
&\quad -\tfrac{1}{4}\overline{u}(p_2)\,[(p_{1\nu}\gamma_\mu + p_{1\mu}\gamma_\nu) + (p_{2\nu}\gamma_\mu + p_{2\mu}\gamma_\nu) \\
&\qquad\qquad\qquad\qquad\qquad +2\left(\eta_{\mu\nu} - \tfrac{q_\mu q_\nu}{q^2}\right) m]u(p_1)
\end{aligned}
$$

This expression is also traceless.

Remarks on the Double Scaling Limit

Geometry by dense Feynman graphs of the same topology ($1/N$ expansion).

Double scaling limit is enforced in

$$Z_N(g) = \int D\hat{\Phi}\ e^{\{-\beta\int d^dx Tr[\hat{\Phi}(x)\hat{K}\hat{\Phi}(x)+V(\hat{\Phi}(x))]\}}$$

$\hat{\Phi}(x)$ - NxN Hermitian matrix

V($\hat{\Phi}$) depends on coupling(s) constant(s) $\{g_i\}$.

Dynamically triangulated random surfaces **summed** on different topologies viewed as the manifold for string propagation ($d = 0$ Matrix models)

Nonperturbative treatment of string theory, when the double scaling limit is enforced ($N \to \infty$ and $g \to g_c$).

Genus (G) expansion of the free energy:

$$F = \ln Z_N \; = \; a \; + \; b \ln \beta + \sum_{G,S} N^{2(1-G)} \; (\frac{N}{\beta})^S \; F_S$$
$$\sim \sum_G (\frac{1}{N})^{2G-2} \mathcal{A}_G\{g_i\}$$

Weak coupling limit, $\frac{1}{N} \sim \frac{1}{\beta} \to 0$ - a one-dimensional frozen Dyson gas, the planar graphs dominate.

Pauli repulsion between the eigenvalues - at strong coupling, at critical point $\{g_i\} = \{g_{iC}\}$ as $N \to \infty$ and $\frac{N}{\beta} \to 1$ (or $\{g_i\} \to \{g_{iC}\}$) non-planar graphs become important.

Factorial growth of the positive $\mathcal{A}_G\{g_i\}$ with the genus G, thus the topological series is not Borel summable and a nonperturbative approach is needed.

At a given topology, $\mathcal{A}_G\{g\}$ has a finite radius of convergence, when expanded in powers of the coupling constant g.

The critical exponents of this model are calculable since the model belongs to the same universality class of two-dimensional, conformally invariant matter field in gravitational background.

A major progress can be achieved if extended to $d > 1$ dimensions. One will then obtain a relation between d dimensional QFT and its possible representation as a stringy object.

$O(N)$ symmetric vector models represent discretized filamentary surfaces - randomly branched polymers in their double scaling limit.

$O(N)$ supersymmetric model at finite temperature

$$\mathcal{Z} = \int [\mathrm{d}\phi][\mathrm{d}R][\mathrm{d}L]\,\mathrm{e}^{-\mathcal{S}_N(\phi,R,L)}$$

$$\begin{aligned}\mathcal{S}_N = \int \mathrm{d}^3x\,\mathrm{d}^2\theta [\tfrac{1}{2}\bar{\mathrm{D}}\phi\mathrm{D}\phi + NU(R) \\ + L(\phi^2 - NR] \\ + \tfrac{1}{2}(N-1)\mathrm{Str}\,\ln\left[-\bar{\mathrm{D}}\mathrm{D} + 2L\right]\end{aligned}$$

$$\begin{aligned}\Delta(k,\theta,\theta) = \frac{1}{k^2 + M_T^2 + \lambda} \\ + \bar{\theta}\theta M_T \left(\frac{1}{k^2 + M_T^2 + \lambda} - \frac{1}{k^2 + M_T^2}\right)\end{aligned}$$

M_T the expectation value of $M(x)$ at finite temperature T.

$$
\begin{aligned}
&G_2(m_T, T) \\
&\qquad = \frac{T}{(2\pi)^{d-1}} \sum_{n\in\mathcal{Z}} \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi n T)^2 + k^2 + m_T^2} \\
&\qquad = \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega(k)} \left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\beta\omega(k)} - 1} \right)
\end{aligned}
$$

Fermions:

$$
\begin{aligned}
&\mathcal{G}_2(M_T, T) \\
&\qquad = \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega(k)} \left(\frac{1}{2} - \frac{1}{\mathrm{e}^{\omega(k)/T} + 1} \right)
\end{aligned}
$$

with $\omega(k) = \sqrt{k^2 + M_T^2}$.

$$
\begin{aligned}
\frac{1}{N}\mathcal{F} = \quad & -\frac{F^2}{2N} + M_T\frac{F\varphi}{N} \\
& + \lambda\frac{\varphi^2}{2N} + \tfrac{1}{2}s(U'(\rho) - M_T) \\
& - \frac{1}{12\pi}\left(m_T^3 - |M_T|^3\right) \\
& + \tfrac{1}{2}\lambda(\rho_c - \rho) \\
+ T\int \frac{\mathrm{d}^2k}{(2\pi)^2} & \left\{\ln[1 - \mathrm{e}^{-\beta\omega_\varphi}] - \ln[1 + \mathrm{e}^{-\beta\omega_\psi}]\right\}
\end{aligned}
$$

Peculiar transitions occur in this system

Region **(II)** $\mu - \mu_c \geq 0$ and $u \geq u_c$: (see Fig. 4 at T=0).

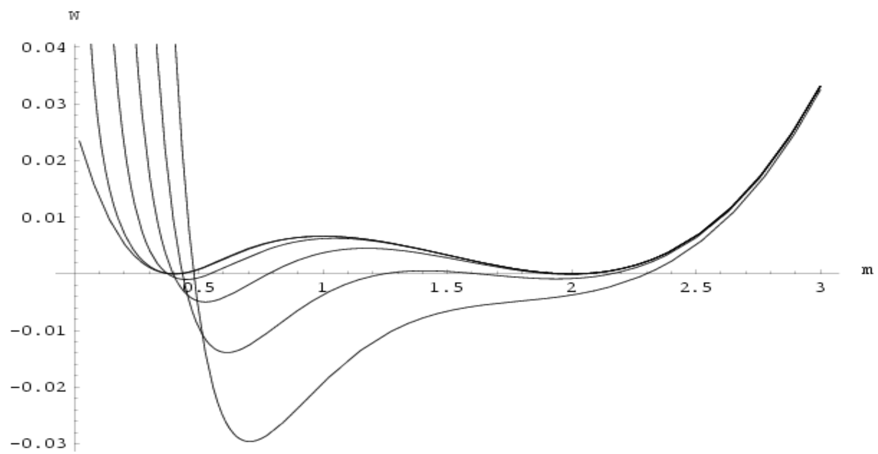


Fig. 5 Ground state free energy at $\varphi = 0$ as a function of the boson mass (m) at different temperatures. Here $\mu - \mu_c = 1$(sets the mass scale) , $u/u_c = 1.5$ and T varies between $T = 0 - -0.5$. At $T = 0$ two degenerate phases with a light $m = m_+$ and heavier $m = m_- > m_+$ boson (and fermion). Light mass phase is affected as temperature increases.

Region **(IV)** $\mu \leq 0$ and $u \leq u_c$:

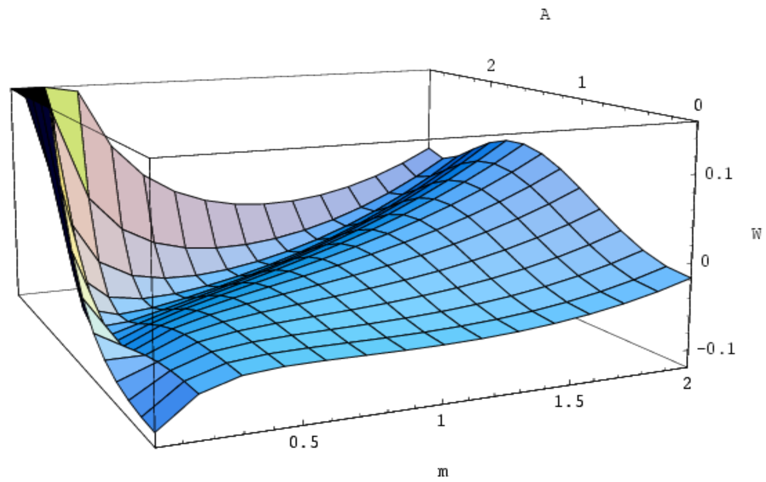


Fig. 6 Same as Fig. 2 but the temperature has been increased from $T = 0$ (in Fig. 2) to $T = 0.7$ (here). $\frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Here $\mu = -1$ and $u/u_c = 0.2$. A non-degenerate $O(N)$ symmetric ground state ($\varphi = 0$) appears with a very small boson mass (the non-zero mass is not seen here due to the limited resolution of the plot).

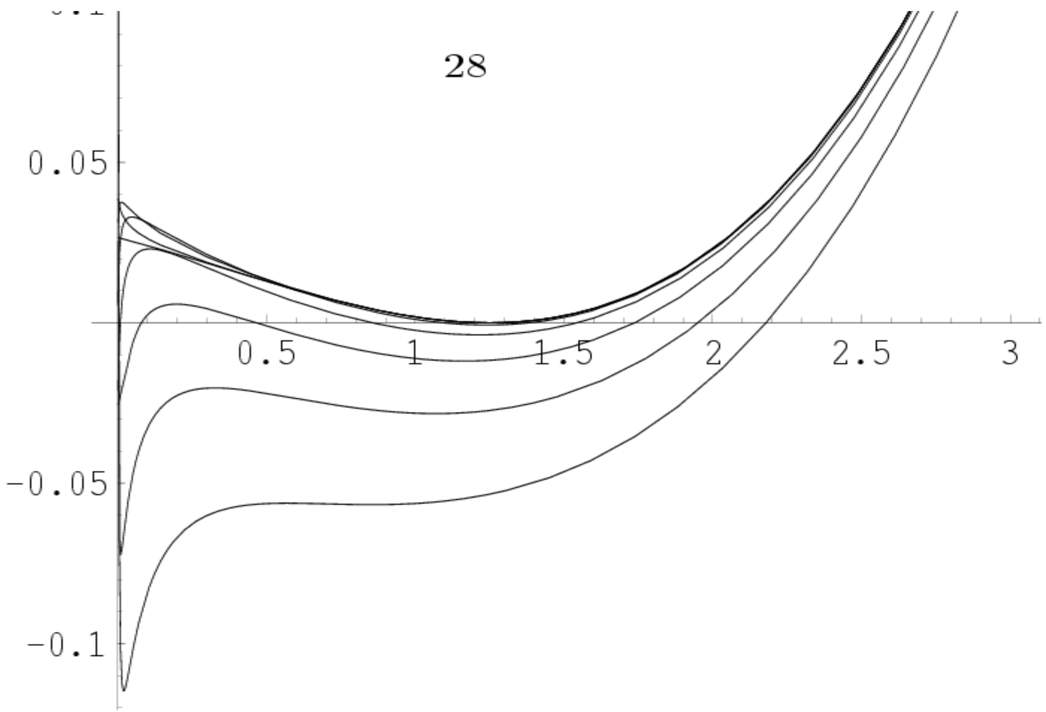


Fig. 7 This figure displays the effect of increasing the temperature from $T = 0$ in Fig. 2 to T that varies between $T = 0 - 0.7$ (Fig. 6 has $T = 0.7$). The ground state free energy $\frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ at $\varphi = 0$ is plotted as a function of the boson mass (m) at different temperatures. Here $\mu = -1$, $u/u_c = 0.2$. At $T = 0$ there are two degenerate phases: An $O(N)$-symmetric phase, shown here, with a massive ($m = m_-$) boson and fermion and an ordered phase ($\varphi \neq 0$) with massless particles (both phases are shown in Fig. 2). At finite temperatures the $O(N)$ symmetry is restored (see Fig. 6) and a small mass ground state appears, the heavy mass state decays into the small mass ground state as seen here.

Supersymmetric $O(N)$ non-linear σ-model at finite temperature

$$\mathcal{Z} = \int [\mathrm{d}\Phi][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}(\Phi,L)}$$

$$\mathcal{S}(\Phi, L) = \frac{1}{2\kappa}\int \mathrm{d}^d x\, \mathrm{d}^2\theta\, \bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi + L(\Phi^2 - N)$$

The free energy is given by

$$\frac{1}{N}\mathcal{F} = \tfrac{1}{2}(M_T^2 - m_T^2)(\frac{1}{\kappa} - \frac{1}{\kappa_c}) - \frac{1}{12\pi}(m_T^3 - |M_T|^3) + T\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\{\ln[1 - \mathrm{e}^{-\beta\omega_\varphi}] - \ln[1 + \mathrm{e}^{-\beta\omega_\psi}]\}$$

(for $\varphi = 0$):

$$\frac{1}{N}\mathcal{F} = \frac{1}{24\pi}(m_T - |M_T|)^2(m_T + 2|M_T|)$$

$$+\frac{T}{4\pi}(m_T^2 - M_T^2)\ln(1-\mathrm{e}^{-m_T/T})$$
$$+T\int\frac{\mathrm{d}^2k}{(2\pi)^2}\{\ln(1-\mathrm{e}^{-\beta\omega_\varphi})-\ln(1+\mathrm{e}^{-\beta\omega_\psi})\}$$

notation

$$X(\kappa,T)=\exp\left[\frac{2\pi}{T}\left(\frac{1}{\kappa_c}-\frac{1}{\kappa}\right)\right].$$

An interesting non-analytic behaviour:

$$\begin{cases} M_T = 0 & \text{for } X < 2, \\ M_T = 2T\ln\left[\frac{1}{2}(X+\sqrt{X^2-4})\right] & \\ & \text{for } X > 2 \end{cases}$$

The boson thermal mass m_T:

$$m_T = 2T\ln\left[\tfrac{1}{2}(X+\sqrt{X^2+4})\right].$$

For $\kappa < \kappa_c$ and $T \to 0$ we find the asymptotic behaviour

$$m_T \sim TX(\kappa,T),$$

Dimension $d=2$. At high temperature

$$\frac{T}{m_T}\sim\frac{1}{\pi}\ln(m_T/m)\sim\frac{1}{\pi}\ln(T/m),$$

Scalar-Fermion thermal mass difference at finite temperature

$$m_A^2 - m_\psi^2 = u[\tfrac{m_\psi}{2\pi}(|m_\psi| - m_A) + \tfrac{m_\psi}{\beta\pi}\ln(\tfrac{1+e^{-\beta|m_\psi|}}{1-e^{-\beta m_A}})]$$

Clearly $m_\varphi^2 \neq m_\psi^2$ at $T \neq 0$

Dilatino mass:

$$M_\psi^D \approx 2\left(1 + \tfrac{u}{u_c}\tfrac{m_\psi}{|m_\psi|}\right) + \tfrac{u}{u_c}\tfrac{\delta}{m_\psi}$$

δ is the boson-fermion thermal mass difference.

$$\frac{1}{N}\left\langle T_{11}\right\rangle_T = \frac{1}{N}\left\langle T_{22}\right\rangle_T =$$
$$- \frac{(m_\varphi - |m_\psi|)^2(m_\varphi + 2|m_\psi|)}{24\pi}$$
$$+ \frac{m_\psi^2 - m_\varphi^2}{4\pi\beta}\ln(1 - e^{-\beta m_\varphi})$$
$$+ \frac{1}{2\pi\beta^3}\int_{\beta|m_\psi|}^{\beta m_\varphi} y\ln(1 - e^{-y})dy$$

$$\frac{1}{N}\left\langle T_{00}\right\rangle_T = -\frac{(m_\varphi - |m_\psi|)^2(m_\varphi + 2|m_\psi|)}{24\pi} + \frac{m_\psi^2 + m_\varphi^2}{4\pi\beta}\ln(1 - e^{-\beta m_\varphi})$$
$$- \frac{1}{\pi\beta^3}\int_{\beta|m_\psi|}^{\beta m_\varphi} y\ln(1 - e^{-y})dy - \frac{m_\psi^2}{2\pi\beta}\ln(1 + e^{-\beta|m_\psi|})$$

and thus the trace of the energy momentum tensor is:

$$\frac{1}{N}\left\langle\ T_\mu{}^{\ \mu}\right\rangle_T = -\frac{(m_\varphi - |m_\psi|)^2(m_\varphi + 2|m_\psi|)}{8\pi} + \frac{3m_\psi^2}{4\pi\beta}\ln(1 - e^{-\beta m_\varphi})$$
$$- \frac{m_\varphi^2}{4\pi\beta}\ln(1 - e^{-\beta m_\varphi}) - \frac{m_\psi^2}{2\pi\beta}\ln(1 + e^{-\beta|m_\psi|})$$

Using the gap equations this simplifies to:

$$\left\langle\ T_\mu{}^{\ \mu}\right\rangle_T = N(m_\psi^2 - m_\varphi^2)\frac{\mu_R}{2u}$$

Taking into account the gap equations, we get the following expression for the thermal expectation value of the energy-momentum trace

$$\left\langle\, T_\mu^{\ \ \mu} \right\rangle_T = N(m_\psi^2 - m_\varphi^2)\tfrac{\mu_R}{2u}$$

Supersymmetry is softly broken when the temperature is turned on but the vanishing of the trace of the energy momentum tensor is guaranteed at $\mu_R = 0$.

Chern-Simons gauge field coupled to a U(N) scalar - light cone gauge

William A. Bardeen and M. M.   JHEP 1406 (2014) 113

$$\mathcal{S}_{\rm CS}(\mathbf{A}) = -\frac{i\kappa}{4\pi}\epsilon_{\mu\nu\rho}\int \mathrm{d}^3x\, Tr\left[\mathbf{A}_\mu(x)\partial_\nu\mathbf{A}_\rho(x) + \frac{2}{3}\mathbf{A}_\mu(x)\mathbf{A}_\nu(x)\mathbf{A}_\rho(x)\right]$$

$$\mathcal{S}_{\rm Scalar} = \int \mathrm{d}^3x \left[(\mathbf{D}_\mu\phi(x))^\dagger\cdot\mathbf{D}_\mu\phi(x) + NV(\phi(x)^\dagger\cdot\phi(x)/N)\right].$$

in the light-cone gauge the action is linear in A^a_+

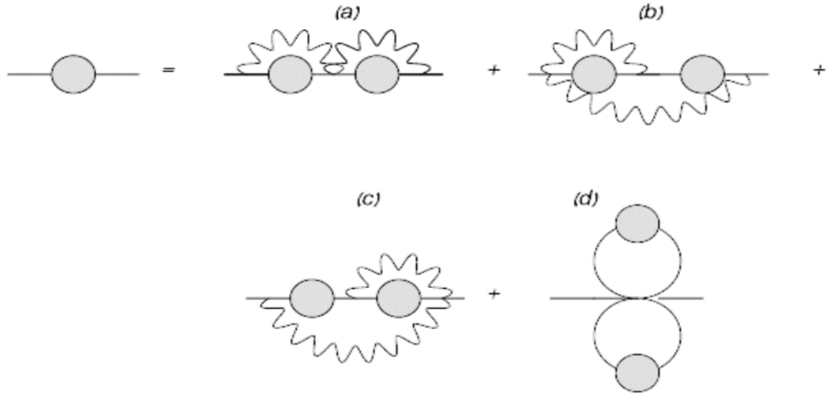
$$\begin{aligned}\mathcal{S}_{\rm CS} + \mathcal{S}_{\rm Scalar} = \int \mathrm{d}^3x \Big\{ & \frac{\kappa}{4\pi}A^a_+\partial_-A^a_3 - \phi^\dagger(\partial_3^2 + 2\partial_+\partial_-)\phi \\ & -\phi^\dagger {A^a}_+ T^a\partial_-\phi + \partial_-\phi^\dagger {A^a}_+T^a\phi \\ & -\phi^\dagger A_3T^a\partial_3\phi + \partial_3\phi^\dagger {A^a}_3T^a\phi \\ & -\phi^\dagger\left({A^a}_3{A^a}_3T^aT^b\right)\phi + NV(\phi^\dagger\cdot\phi/N)\Big\}\end{aligned}$$

$$-\frac{\kappa}{4\pi}\partial_- A_3^a = J_-^a = \phi^\dagger T^a \partial_- \phi - \partial_- \phi^\dagger T^a \phi$$

$$A_3^a(p) = \left(\frac{2\pi}{\kappa}\right)\frac{2ip^+}{{p^+}^2+\epsilon^2}J_-^a \quad {}_{\epsilon\to 0} \to \left(\frac{4\pi i}{\kappa}\right)\frac{1}{p^+}J_-^a$$

$$G_{+3}(p) = -G_{3+}(p) = \frac{4\pi i}{\kappa}\frac{1}{p^+} = 4\pi i \frac{\lambda}{N}\frac{1}{p^+}$$

$$NV(\phi^\dagger\cdot\phi/N) = \mu^2\phi^\dagger\cdot\phi + \frac{1}{2}\frac{\lambda_4}{N}(\phi^\dagger\cdot\phi)^2 + \frac{1}{6}\frac{\lambda_6}{N^2}(\phi^\dagger\cdot\phi)^3$$



$$\Sigma^{(a,b,c)}(p,\lambda)_{ij} = \delta_{ij}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\int\frac{\mathrm{d}^3l}{(2\pi)^3}\Bigg\{4\pi^2\lambda^2\frac{(l+p)^+(q+p)^+}{(l-p)^+(q-p)^+}\left(\frac{1}{(q^2+\Sigma(q))(l^2+\Sigma(l))}\right)$$
$$-8\pi^2\lambda^2\frac{(l+p)^+(q+l)^+}{(l-p)^+(q-l)^+}\left(\frac{1}{(q^2+\Sigma(q))(l^2+\Sigma(l))}\right)\Bigg\} \tag{2.8}$$

which sum up to

$$\Sigma^{(a,b,c)}(p,\lambda)_{ij} = 4\pi^2\lambda^2\delta_{ij}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\int\frac{\mathrm{d}^3l}{(2\pi)^3}\frac{1}{(q^2+\Sigma(q))(l^2+\Sigma(l))}$$

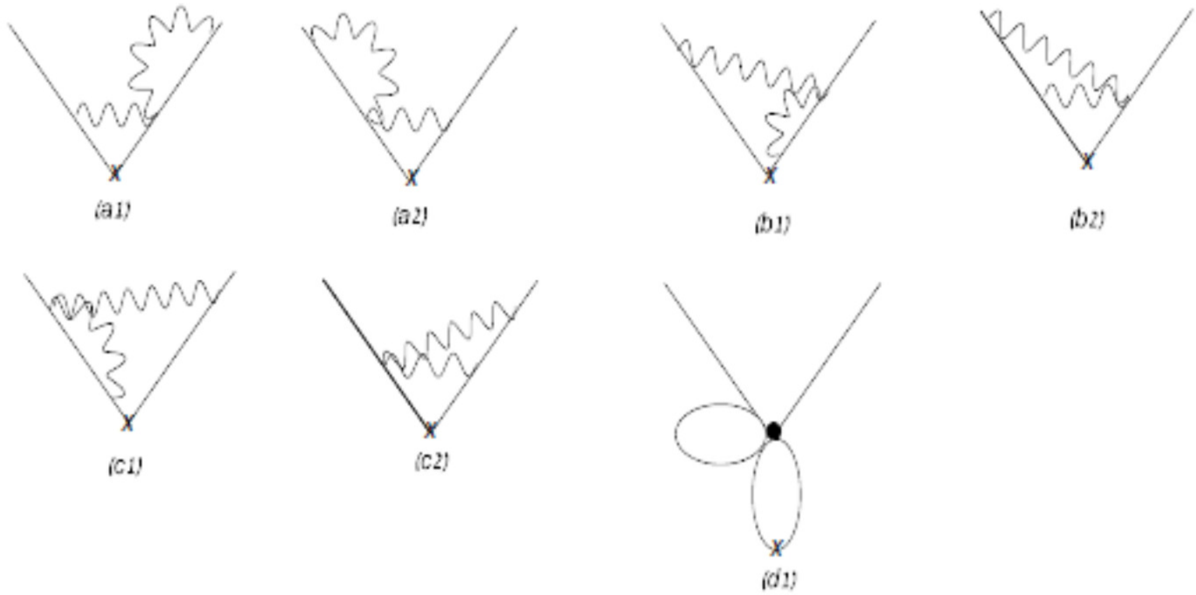
$$\Sigma^{(d)}(p,\lambda)_{ij} = \frac{1}{2}\lambda_6\delta_{ij}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\int\frac{\mathrm{d}^3l}{(2\pi)^3}\frac{1}{(q^2+\Sigma(q))(l^2+\Sigma(l))}$$

$$\Sigma(p,\lambda,\lambda_6) = 4\pi^2\left(\lambda^2+\frac{\lambda_6}{8\pi^2}\right)\left\{\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{(q^2+\Sigma(q))}\right\}^2$$

$$\Sigma(p, \lambda, \mu, \lambda_4 \lambda_6) = \frac{1}{4}\left(\lambda^2 + \frac{\lambda_6}{8\pi^2}\right)|\Sigma| - \lambda_{4R}\frac{\sqrt{|\Sigma|}}{4\pi} + \mu_R^2$$

$$\Sigma = \frac{1}{4}\left(\lambda^2 + \frac{\lambda_6}{8\pi^2}\right)|\Sigma|$$

(a) $\boldsymbol{\Sigma = M^2 = 0}$ (b) $\boldsymbol{\Sigma} = M^2 \neq 0$ if $\lambda^2 + \frac{\lambda_6}{8\pi^2} = 4$



$$V^{(a-d)}(p,k_3) = V = -8\pi^2\left(\lambda^2+\frac{\lambda_6}{8\pi^2}\right)\int\frac{d^3l}{(2\pi)^3}\left(\frac{1}{l^2+\Sigma}\right)\int\frac{d^3q}{(2\pi)^3}\left(\frac{1}{(l+k)^2+\Sigma}\right)\left(\frac{1}{q^2+\Sigma}\right)$$

The contribution to the vertex of diagrams a1-2, b1-2 and c1-2 in figure 3 is:

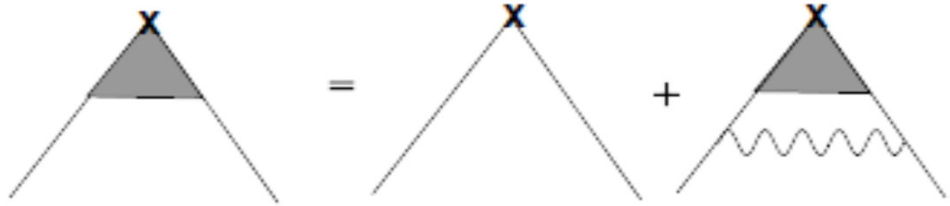
$$V^{(a1-2,b1-2,c1-2)}(p,k) = \lambda^2 4\pi^2 \int \frac{d^3l}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}$$
$$\Bigg\{ \; -\left(\frac{l+p}{l-p}\right)^{+}\left(\frac{q+p+k}{q-p}\right)^{+}\left(\frac{1}{l^2+\Sigma}\right)\left(\frac{1}{(l+k)^2+\Sigma}\right)\left(\frac{1}{(q+k)^2+\Sigma}\right)$$
$$-\left(\frac{l+p}{l-p}\right)^{+}\left(\frac{q+p+k}{q-p}\right)^{+}\left(\frac{1}{l^2+\Sigma}\right)\left(\frac{1}{q^2+\Sigma}\right)\left(\frac{1}{(q+k)^2+\Sigma}\right)$$
$$+\left(\frac{l+p}{l-p}\right)^{+}\left(\frac{q+l+2k}{q-l}\right)^{+}\left(\frac{1}{l^2+\Sigma}\right)\left(\frac{1}{(l+k)^2+\Sigma}\right)\left(\frac{1}{(q+k)^2+\Sigma}\right)$$
$$+\left(\frac{l+p}{l-p}\right)^{+}\left(\frac{l+q}{q-l}\right)^{+}\left(\frac{1}{l^2+\Sigma}\right)\left(\frac{1}{q^2+\Sigma}\right)\left(\frac{1}{(q+k)^2+\Sigma}\right)$$
$$+\left(\frac{l+q+k}{l-q}\right)^{+}\left(\frac{q+p+2k}{q-p}\right)^{+}\left(\frac{1}{l^2+\Sigma}\right)\left(\frac{1}{(l+k)^2+\Sigma}\right)\left(\frac{1}{(q+k)^2+\Sigma}\right)$$
$$+\left(\frac{l+q}{l-q}\right)^{+}\left(\frac{q+p+k}{q-p}\right)^{+}\left(\frac{1}{l^2+\Sigma}\right)\left(\frac{1}{q^2+\Sigma}\right)\left(\frac{1}{(q+k)^2+\Sigma}\right)\Bigg\} \tag{2.19}$$

The self interaction of the scalar fields contributes to the vertex the term

$$V^{(d1-2)}(p,k) = -\frac{1}{2}\lambda_6 \int \frac{d^3l}{(2\pi)^3}\frac{d^3q}{(2\pi)^3} \quad \Bigg\{ \left(\frac{1}{l^2+\Sigma}\right)\left(\frac{1}{(l+k)^2+\Sigma}\right)\left(\frac{1}{q^2+\Sigma}\right)$$
$$+\left(\frac{1}{l^2+\Sigma}\right)\left(\frac{1}{q^2+\Sigma}\right)\left(\frac{1}{(q+k)^2+\Sigma}\right)\Bigg\} \tag{2.20}$$

When all vertex contributions are added at $k^+ = 0$, diagrams a1-2, b1-2, c1-2, d1-2 result in:

$$V^{(a-d)}(p,k_3) = V = -8\pi^2\left(\lambda^2 + \frac{\lambda_6}{8\pi^2}\right)\int \frac{d^3l}{(2\pi)^3}\left(\frac{1}{l^2+\Sigma}\right)\int \frac{d^3q}{(2\pi)^3}$$
$$\left(\frac{1}{(l+k)^2+\Sigma}\right)\left(\frac{1}{q^2+\Sigma}\right) \tag{2.21}$$

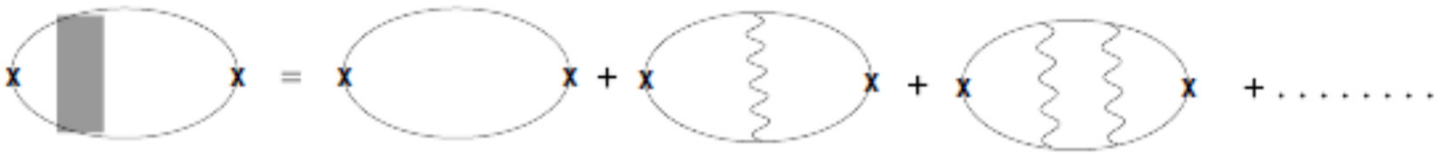


$$V(p^2,k_3) = 1 + i4\pi\lambda k_3 \int \frac{d^3l}{(2\pi)^3} V(l^2,k_3) \frac{(l+p)^+}{(l-p)^+} \frac{1}{l^2+\Sigma} \frac{1}{(l+k)^2+\Sigma}$$

$$V(p^2,k_3) = C \exp\left\{ i\lambda k_3 \int dx (p^2 + x(1-x)k_3^2 + M^2)^{-1/2} \right\}$$

$$= 2 \exp\left\{ i\lambda k_3 \int_0^1 dx (p^2 + x(1-x)k_3^2 + M^2)^{-1/2} \right\}$$

$$\left\{ 1 + \exp[i\lambda k_3 \int_0^1 (x(1-x)k_3^2 + M^2)^{-\frac{1}{2}}] \right\}^{-1}$$



$$B_{CS}(k_3) = \int \frac{d^3 l}{(2\pi)^3} V(l^2, k_3) \left(\frac{1}{(l^2 + l_3^2 + \Sigma} \right) \left(\frac{1}{l^2 + (l_3 + k_3)^2 + \Sigma} \right)$$

$$= \frac{1}{4\pi\lambda} \, \frac{1}{k_3} \tan \left\{ \frac{1}{2} \lambda k_3 \int dx (x(1-x) k_3^2 + M^2)^{-1/2} \right\}$$

$$= \frac{1}{4\pi\lambda} \, \frac{1}{k_3} \tan \left\{ \lambda \ \arctan \left(\frac{k_3}{2M} \right) \right\}$$

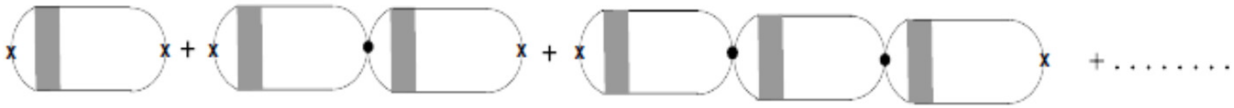


Figure 6. Full planar "bubble graph".

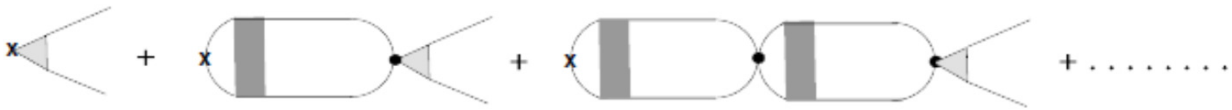


Figure 7. Full planar vertex.

$$W(p^2,k_3) = \quad V(p^2,k_3)(1+\lambda_4^{\text{eff}} B_{CS}(k_3))^{-1}$$

$$\lambda_4^{\text{eff}} = \lambda_4 + \left(\lambda^2 + \frac{\lambda_6}{8\pi^2}\right)\frac{8\pi^2}{N}\langle\phi\dagger\cdot\phi\rangle$$

$$\lambda_4^{\text{eff}} = \lambda_{4R} - 2\pi M\left(\lambda^2 + \frac{\lambda_6}{8\pi^2}\right) = -8\pi M$$

$\langle J_0 J_0 \rangle$ correlator and the dilaton

$$\langle J_0(k)\ J_0(-k)\rangle\ = \frac{N}{8\pi M}\left\{1-\frac{k^2}{12M^2}(1-\lambda^2)+\ldots\right\}$$
$$\left\{1+\left(\frac{\lambda_4^{\rm eff}}{8\pi M}\right)\left(1-\frac{k^2}{12M^2}(1-\lambda^2)+\ldots\right)\right\}^{-1}$$

$$= \frac{N}{8\pi M}\left\{1-\frac{k^2}{12M^2}(1-\lambda^2)+\ldots\right\}$$
$$\left\{1+\left(\frac{\lambda_4^{\rm eff}}{8\pi M}\right)\left(1-\frac{k^2}{12M^2}(1-\lambda^2)+\ldots\right)\right\}^{-1}.$$

$$\langle J_0(k)\ J_0(-k)\rangle = \frac{3N}{2\pi}\left(\frac{M}{1-\lambda^2}\right)\frac{1}{k^2} = \frac{f_D^2}{k^2}$$

where

$$f_D = \sqrt{\frac{3NM}{2\pi(1-\lambda^2)}}$$

the effective Lagrangian of the dilaton

In terms of the dilaton field $D(x)$ (where $J_0(x) = f_D D(x)$)

$$\mathcal{L} = \frac{1}{2}\partial_\mu D \cdot \partial_\mu D - g_D(\phi^\dagger \cdot \phi)D$$

where $g_D = -\frac{M^{3/2}}{\sqrt{N}}\sqrt{(96\pi)/(1-\lambda^2)}$

Explicit breaking of scale invariance and the pseudo-dilaton

$$\frac{1}{4}\left(\lambda^2 + \frac{\lambda_6}{8\pi^2}\right) = 1 - \delta$$

$$M^2\delta = -\lambda_{4R}\left(\frac{M}{4\pi}\right) + \mu_R^2$$

$$1 + \lambda_4^{\text{eff}} B_{CS}(k)$$

$$= \left(\frac{\lambda_{4R}}{8\pi M}\right) + \delta + \left(\frac{1}{12}\right)\left(\frac{k^2}{M^2}\right)(1-\lambda^2) + \ldots$$

can read off the mass of the pseudo-dilaton

$$M_{pD}^2 = \left(\frac{12M^2}{(1-\lambda^2)}\right)\left[\left(\frac{\lambda_{4R}}{8\pi M}\right) + \delta\right]$$

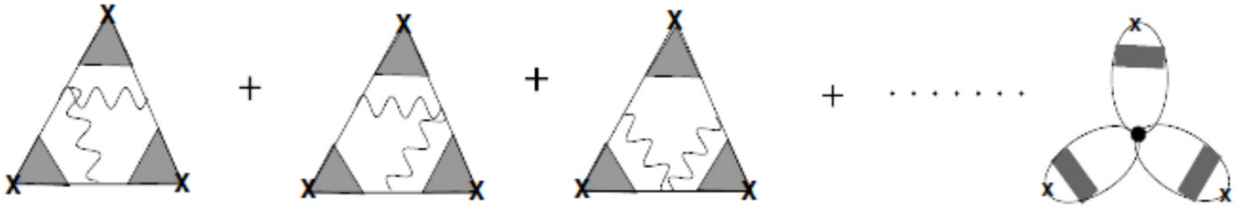


Figure 8. $\langle J_0(k)J_0(k')\ J_0(-k-k')\rangle$.

$$V_e = -\left(\lambda^2 + \frac{\lambda_6}{8\pi^2}\right)\frac{N}{64\pi}\frac{1}{M^3}\left\{1 + \frac{(\lambda^2-1)}{12M^2}(k_3^2 + k_3'^2 + (k_3+k_3')^2) + \mathcal{O}\left(\frac{(k_3,k_3')^4}{M^4}\right)\right\}$$

$$V_{\text{massive phase}} = -\frac{N}{16\pi}\frac{1}{24M^5}(k_3^2 + k_3'^2 + (k_3+k_3')^2) + \mathcal{O}\left(\frac{(k_3,k_3')^4}{M^7}\right)$$

$$\langle J_0(k)J_0(k')\ J_0(-k-k')\rangle = V(k,k',-k-k')(1+\lambda_4^{\text{eff}}B_{CS}(k))^{-1}(1+\lambda_4^{\text{eff}}B_{CS}(k'))^{-1}(1+\lambda_4^{\text{eff}}B_{CS}(-k-k'))^{-1}$$

$$= -\frac{9N}{2\pi}\frac{M}{(1-\lambda^2)^3}\left\{\frac{1}{k^2k'^2} + \frac{1}{k^2(k+k')^2} + \frac{1}{k'^2(k+k')^2}\right\} \quad + \mathcal{O}\left(\frac{M^3}{k^2}\right)$$

The dilaton self interaction can be now defined in the effective Lagrangian

$$\mathcal{L}_3D = g_{3D}(\partial_\mu D \cdot \partial_\mu D)D \qquad\qquad g_{3D} = -\sqrt{\frac{6\pi}{NM(1-\lambda^2)^3}}$$

3D field theories with Chern-Simons term for large N in the Weyl gauge

M. M and Jean Zinn-Justin **JHEP 1501 (2015) 054**

$$\mathcal{S}(\psi, \bar{\psi}, \mathbf{A}) = \mathcal{S}_{\mathrm{CS}}(\mathbf{A}) + \mathcal{S}_{\mathrm{F}}(\psi, \bar{\psi}, \mathbf{A})$$

We now add to the Chern-Simons action, quantized in the $\mathbf{A}_3 = 0$ gauge, a $\mathrm{U}(N)$ gauge-invariant action for an N-component spinor field ψ,

$$\mathcal{S}_{\mathrm{CS}}(\mathbf{A}) = \frac{N}{ig}\mathrm{CS}_3(\mathbf{A}) = \frac{N}{ig}\int \mathrm{d}^3x \,\mathrm{tr}\left[\mathbf{A}_2(x)\partial_3\mathbf{A}_1(x) - \mathbf{A}_1(x)\partial_3\mathbf{A}_2(x)\right]$$

with $$\mathcal{S}_{\mathrm{F}}(\psi, \bar{\psi}, \mathbf{A}) = -\int \mathrm{d}^3x\, \bar{\psi}(x)(\not{\mathbf{D}} + M_0)\psi(x)$$

Integrating out the gauge field $$\ln \mathcal{Z} = (2\pi)^3 \frac{g}{2N}\int \mathrm{d}^3p\, \tilde{J}_2^a(-p)\frac{1}{p_3}\tilde{J}_1^a(p)$$

gauge field propagator $$\tilde{\Delta}^{ab}_{\alpha\beta}(p) = \epsilon_{\alpha\beta}\delta^{ab}\frac{g}{2N}\frac{1}{p_3}$$

$$
\begin{aligned}
\mathcal{S} = &-(2\pi)^3 \int \mathrm{d}^3 p\, \bar{\psi}(p) \cdot (i\not{p} + M_0)\psi(p) \\
&- \frac{ig}{N}(2\pi)^3 \int \mathrm{d}^3 p\, \mathrm{d}^3 p' \mathrm{d}^3 q\, \mathrm{d}^3 q'\, \delta^{(3)}(p + q - p' - q') \\
&\times \bar{\psi}_1(p) \cdot \psi_1(p') \mathrm{PP}\frac{1}{q_3 - p_3'} \bar{\psi}_2(q) \cdot \psi_2(q').
\end{aligned}
$$

The large N action

additional bilocal (in Euclidean time) composite fields $\{\rho_\alpha(t', t, x)\}$ and $\{\lambda_\alpha(t, t', x)\}$.

$$
\begin{aligned}
\mathcal{S} = &- \int \mathrm{d}t\, \mathrm{d}^2 x\, \bar{\psi}(t, x) \cdot (\not{\partial} + M_0)\psi(t, x) \\
&+ \frac{1}{2} gN \int \mathrm{d}^2 x\, \mathrm{d}t\, \mathrm{d}t' \mathrm{sgn}(t' - t) \rho_1(t', t, x) \rho_2(t, t', x) \\
&+ \int \mathrm{d}^2 x\, \mathrm{d}t\, \mathrm{d}t' \sum_{\alpha=1}^{2} \lambda_\alpha(t, t', x) \left[N \rho_\alpha(t', t, x) - \bar{\psi}_\alpha(t, x) \cdot \psi_\alpha(t', x) \right]
\end{aligned}
$$

$$\mathcal{S}_N/N = -\mathrm{tr}\ln\mathbf{K} + \int \mathrm{d}^2x\,\mathrm{d}t\,\mathrm{d}t'\left[\sum_{\alpha=1}^{2}\lambda_\alpha(t,t',x)\rho_\alpha(t',t,x) + \frac{1}{2}g\mathrm{sgn}(t'-t)\rho_1(t',t,x)\rho_2(t,t',x)\right]$$

$$K_{\alpha\beta}(t,x;t',x') = \left(\partial\!\!\!/_{\alpha\beta} + \delta_{\alpha\beta}M_0\right)\delta(t-t')\delta^{(2)}(x-x') + \delta_{\alpha\beta}\lambda_\alpha(t,t',x)\delta^{(2)}(x-x').$$

Saddle point $$-\rho_\alpha(t,t',x) + [\mathbf{K}^{-1}]_{\alpha\alpha}(t,x;t',x) = 0\,,\ \ \alpha = 1,2\,.$$

$$\lambda_1(t) = \frac{1}{2}g\mathrm{sgn}(t)\rho_2(t)\ ,\ \ \lambda_2(t) = -\frac{1}{2}g\mathrm{sgn}(t)\rho_1(t)$$

$$\tilde{\mathbf{K}}(\omega,p) = i\omega\sigma_3 + ip\!\!\!/ + M_0 + \pi\left(\tilde{\lambda}_1(\omega) + \tilde{\lambda}_2(\omega)\right) + \pi\left(\tilde{\lambda}_1(\omega) - \tilde{\lambda}_2(\omega)\right)\sigma_3$$

$$\mu_1(\omega) = M_0 + i\omega - ig\int \frac{\mathrm{d}\omega'}{\omega - \omega'}\tilde{\rho}_2(\omega')$$
$$\mu_2(\omega) = M_0 - i\omega + ig\int \frac{\mathrm{d}\omega'}{\omega - \omega'}\tilde{\rho}_1(\omega')$$

The four equations can be summarized by the unique pair of equations

$$\mu_2(\omega) = M_0 - i\omega + ig\int \frac{\mathrm{d}\omega'}{\omega - \omega'}\tilde{\rho}_1(\omega')\,,$$
$$\tilde{\rho}_1(\omega) = \frac{\mu_2(\omega)}{(2\pi)^3}\int \frac{\mathrm{d}^2 k}{k^2 + |\mu_2(\omega)|^2}\,.$$

solution

$$\mu_2(\omega) = M_0 - i\omega - \frac{ig}{(2\pi)^3}\int \mathrm{d}^3 p \frac{(M - ip_3)}{(p_3 - \omega)\,(p^2 + M^2)}\exp[ig\Theta(p_3)]$$

where $$\Theta(\omega) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p}{(\omega - p_3)\,(M^2 + p^2)} = \frac{1}{4\pi}\arctan\left(\frac{\omega}{M}\right)$$

$$\mu_2(\omega) = M_0 - M - g\Omega_1(M) + (M - i\omega)\exp[ig\Theta(\omega)]$$

We then choose the mass parameter M to be the solution of the *gap equation*

$$M_0 = M + g\Omega_1(M)$$

Finally $$|\mu_2(\omega)|^2 = M^2 + \omega^2$$

The free energy density

$$W = \frac{1}{NV} \ln(\mathcal{Z}/\mathcal{Z}_0)$$

$$\begin{aligned} W = {} & \frac{1}{(2\pi)^3} \int \mathrm{d}^3 p \, \ln(1 + M^2/p^2) \\ & + ig \int \frac{\mathrm{d}\omega \, \mathrm{d}\omega'}{\omega - \omega'} (M - i\omega)(M + i\omega') \mathrm{e}^{ig[\Theta(\omega) - \Theta(\omega')]} \\ & \times \frac{1}{(2\pi)^6} \int \frac{\mathrm{d}^2 p \, \mathrm{d}^2 p'}{(p^2 + \omega^2 + M^2)\,(p'^2 + \omega'^2 + M^2)}. \end{aligned}$$

The fermion two-point function for N large

$$\langle \psi^i_\alpha(x)\bar{\psi}^j_\beta(x')\rangle_0 = \delta_{ij} W^{(2)}_{\alpha\beta}(x-x')$$

In the large N limit, the fermion two-point function is obtained by inverting

$$K_{\alpha\beta}(t,x;t',x') = \left(\not{\partial}_{\alpha\beta} + \delta_{\alpha\beta} M_0\right)\delta(t-t')\delta^{(2)}(x-x') + \delta_{\alpha\beta}\lambda_\alpha(t,t',x)\delta^{(2)}(x-x').$$

$$\tilde{\mathbf{K}}^{-1} = -\frac{ip_1\sigma_1 + ip_2\sigma_2 + (i\omega\sigma_3 - M)\exp\left[ig\sigma_3\Theta(\omega)\right]}{p^2+M^2}$$

$$\tilde{W}^{(2)}(p) = -\tilde{\mathbf{K}}^{-1}(p) = U(p_3)\frac{(i\not{p} - M)}{p^2+M^2}U(p_3)$$

where $$U(\omega) = \exp\left[\frac{1}{2} ig\sigma_3\Theta(\omega)\right]$$

$$\tilde{\Gamma}^{(2)}(p) = -U^{-1}(p_3)\left(i\not{p} + M\right)U^{-1}(p_3).$$

Gauge-invariant observables

No summation
$$R_\alpha(x) = \frac{1}{N}\bar{\psi}_\alpha(x)\cdot\psi_\alpha(x)$$

$$R(x) = R_1(x) + R_2(x) \qquad\qquad J_3(x) = i\big(R_1(x) - R_2(x)\big).$$

The equal-time expectation value of ρ_1 is given by

$$\langle R_1(x)\rangle = \int \mathrm{d}\omega\,\tilde{\rho}_1(\omega) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3p\frac{M - ip_3}{p^2+M^2}\exp[ig\Theta(p_3)]$$

$$\langle R\rangle = \langle R_1 + R_2\rangle = 2\langle\rho_1\rangle = 2M\Omega_1(M) + g\Omega_1^2(M).$$

$$= g\frac{\Lambda^2}{16\pi^2} + \frac{\Lambda M}{2\pi}\left(1 - \frac{g}{4\pi}\right) - \frac{M^2}{2\pi}\left(1 - \frac{g}{8\pi}\right)$$

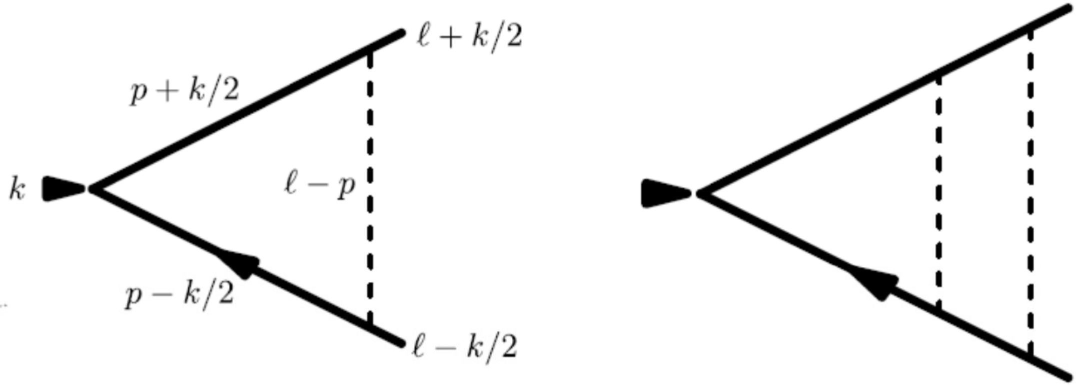
Connected R correlation function at zero momentum

$$\langle \tilde{R}(0)\tilde{R}(0)\rangle_{\rm c} = 2\Omega_1(M) - \frac{4M^2\Omega_2(M)}{1-2gM\Omega_2(M)} = \frac{\Lambda}{2\pi} - \frac{M}{\pi}\frac{(1-g/8\pi)}{(1-g/4\pi)}$$

$$\langle \tilde{R}(0)\tilde{R}(0)\tilde{R}(0)\rangle_{\rm c} = -\frac{1}{\pi}\frac{(1-g/8\pi)}{(1-g/4\pi)^2}$$

The $\langle(\bar\psi\psi)\psi\bar\psi\rangle$ vertex function

$$W^{(1,2)}(x;y,z) = \langle \bar\psi(x)\cdot\psi(x)\psi(y)\cdot\bar\psi(z)\rangle_{\mathrm{c}}$$



$$\tilde W^{(1,2)}\left(k;\ell-\frac{1}{2}k,\ell+\frac{1}{2}k\right) = -\tilde W^{(2)}\left(\ell-\frac{1}{2}k\right)\tilde\Gamma^{(1,2)}\left(k;\ell-\frac{1}{2}k,\ell+\frac{1}{2}k\right)\tilde W^{(2)}\left(\ell+\frac{1}{2}k\right)$$

$$\tilde\Gamma^{(1,2)}(k;\ell-k/2,\ell+k/2) = -E(k;\ell-k/2,\ell+k/2) - i\sigma_3 F(k;\ell-k/2,\ell+k/2)$$

at zero momentum:

$$E(0;\ell,\ell) = \frac{1}{1-2gM\Omega_2(M)}U^{-2}(\ell_3)\left(1-\frac{g}{4\pi}\frac{\ell_3^2}{\ell_3^2+M^2}\right)$$

$$F(0;\ell,\ell) = \frac{1}{1-2gM\Omega_2(M)}U^{-2}(\ell_3)\frac{g}{4\pi}\frac{M}{\ell_3^2+M^2}\,.$$

The perturbative expansion of the $\langle(\bar\psi\psi)\psi\bar\psi\rangle$ vertex function at two loops

. . . . . . .and more

$$\tilde V^{(1,2)}(k;\ell-k/2,\ell+k/2)=A(\ell_3,k)+i\sigma_3 B(\ell_3,k)$$

$$A=\sum_{n=0}A_n g^n,\ \ B=\sum_{n=0}B_n g^n,$$

define

$$\Xi(\omega,k)=\frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3p}{(\omega-p_3)\left[(p+k/2)^2+M^2\right]\left[(p-k/2)^2+M^2\right]}$$

$$A(\ell_3,k)=\frac{\cos\left(2g\tau\Xi(\ell_3,k)\right)-(M\ell_3/\tau)\sin\left(2g\tau\Xi(\ell_3,k)\right)}{\cos\big(gk\mathcal{B}_1(k)\big)-2(M/k)\sin\big(gk\mathcal{B}_1(k)\big)}$$

$$B(\ell_3,k)=\frac{t_1}{\tau}\frac{\sin\left(2g\tau\Xi(\ell_3,k)\right)}{\cos\big(gk\mathcal{B}_1(k)\big)-2(M/k)\sin\big(gk\mathcal{B}_1(k)\big)}.$$

R two-point function and vertex three-point function

$$\langle \tilde{R}(k)\tilde{R}(-k)\rangle_c = 2\Omega_1(M) - \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3q\left[\left(k^2+4M^2\right)A(q_3,k)+4Mq_3B(q_3,k)\right]}{\left[(q+k/2)^2+M^2\right]\left[(q-k/2)^2+M^2\right]}$$

simplifies to :

$$\langle \tilde{R}(k)\tilde{R}(-k)\rangle_c = 2\Omega_1(M) - \frac{k^2+4M^2}{kg}\frac{\tan\bigl(gk\mathcal{B}_1(k)\bigr)}{1-2M\tan\bigl(gk\mathcal{B}_1(k)\bigr)/k}$$

where $$gk\mathcal{B}_1(k) = \frac{g}{4\pi}\arctan(k/2M)$$

Mass gap and critical coupling

$$M_0 = M + g\Omega_1(M) = M_c + m$$

where M is the fermion physical mass

$$M_c = g\frac{\Lambda}{4\pi}$$

For $g \neq 4\pi$, the fermion mass, solution of the gap equation, is

$$M = \frac{m}{1 - g/4\pi} .$$

For the special value $m = 0$ or $M_0 = M_c$: $\quad M = \frac{g}{4\pi}|M|$

But ! $$\langle \tilde{R}(k)\tilde{R}(-k)\rangle_c \underset{g\to 4\pi}{\sim} -\frac{1}{(4\pi - g)}\frac{k}{\arctan(k/2M)}$$

Adding a deformation to the Chern-Simons fermion action

$$\mathcal{S}_\sigma = \int \mathrm{d}^3x \left[-\sigma(x)\bar{\psi}(x)\cdot\psi(x) + \frac{N}{3g_\sigma}\sigma^3(x) - N\mathcal{R}\sigma(x) \right]$$

where the new parameters g_σ and $\mathcal{R}$ are fixed when $N \to \infty$.

For $\mathcal{R} \neq 0$, in the classical limit $\sigma(x)$ has a non-vanishing expectation value σ,

$$\sigma^2 = g_\sigma \mathcal{R}$$

$$\sigma(x) = \sigma + \varsigma(x) \qquad \mathcal{S}_\sigma = \int \mathrm{d}^3x \left[-(\sigma + \varsigma(x))\bar{\psi}(x)\cdot\psi(x) + \frac{N}{3g_\sigma}\varsigma^3(x) + \frac{N\sigma}{g_\sigma}\varsigma^2(x) \right]$$

Gap equation:

$$\left[1 - \frac{(g-g_\sigma)}{2\pi}\left(1 - \frac{g}{8\pi}\right)\right] M^2 + \left(1 - \frac{g}{4\pi}\right)\left[\frac{(g-g_\sigma)}{2\pi}\Lambda - 2M_0\right] M$$
$$- g_\sigma\mathcal{R} + M_0^2 - \frac{g}{2\pi}M_0\Lambda + g(g-g_\sigma)\frac{\Lambda^2}{16\pi^2} = 0\,,$$

$$M_0 = M_c + m$$

$$M_c = (g - g_\sigma)\frac{\Lambda}{4\pi} \qquad\qquad \mathcal{R} = \mathcal{R}_c - \frac{m\Lambda}{2\pi} + \frac{\eta}{4\pi}m^2,$$

where η is a constant parameter and

$$\mathcal{R}_c = (g_\sigma - g)\frac{\Lambda^2}{16\pi^2}$$

The gap equation then reads

$$\left[1 - \frac{(g - g_\sigma)}{2\pi}\left(1 - \frac{g}{8\pi}\right)\right] M^2 - 2\left(1 - \frac{g}{4\pi}\right) mM + m^2\left(1 - \eta\frac{g_\sigma}{4\pi}\right) = 0$$

Finally, the gap equation is satisfied for any value of $M \geq 0$ if the coefficient of M^2 also vanishes, that is, for

$$\left(\frac{g - g_\sigma}{2\pi}\right)\left(1 - \frac{g}{8\pi}\right) = 1 \;\Leftrightarrow\; g_\sigma = -\frac{(4\pi - g)^2}{8\pi - g}\,.$$

The generic R two-point function

$$\langle \tilde{R}(k)\tilde{R}(-k)\rangle_{\rm c} = -\frac{2\sigma}{g_\sigma} + \frac{4\sigma^2/g_\sigma^2}{2\sigma/g_\sigma - \langle \tilde{R}(k)\tilde{R}(-k)\rangle_{\rm c,0}}$$

$$\langle \tilde{\varsigma}(k)\tilde{\varsigma}(-k)\rangle = \frac{1}{2\sigma/g_\sigma - \langle \tilde{R}(k)\tilde{R}(-k)\rangle_{\rm c,0}} \equiv \mathcal{D}^{-1}(k)$$

$$\mathcal{D}(k) = -\frac{2m}{g_\sigma} + \frac{2M}{g_\sigma}\left(1 - \frac{g - g_\sigma}{4\pi}\right) + \frac{k^2 + 4M^2}{kg}\frac{\tan\big(gk\mathcal{B}_1(k)\big)}{1 - 2M\tan\big(gk\mathcal{B}_1(k)\big)/k}$$

If in addition $m = 0$, M is undetermined and the two-point function has the form

$$\langle \tilde{\varsigma}(k)\ \tilde{\varsigma}(-k)\rangle \underset{k\to 0}{\sim} \frac{24\pi(4\pi - g)}{(8\pi - g)}\frac{M}{k^2}\,.$$

The dilaton effective action

$$\varsigma(x) = f_D^{-1} D(x)$$

$$\langle \tilde{\varsigma}(k)\ \tilde{\varsigma}(-k) \rangle \underset{k \to 0}{\sim} \frac{f_D^2}{k^2}, \quad f_D = \sqrt{24\pi} \sqrt{\frac{1 - g/4\pi}{2 - g/4\pi}} \sqrt{M}\,.$$

$$\mathcal{S}(D) = \frac{1}{2} \int \mathrm{d}^3 x\, \partial_\mu D(x) \partial_\mu D(x) F(D(x)) + \mathcal{O}(\text{higher order terms in } \partial_\mu)\,.$$

Spontaneously broken scale invariance in boson and fermion theories

The condition we found for the existence of a massive phase

$$\left(\frac{g-g_\sigma}{2\pi}\right)\left(1-\frac{g}{8\pi}\right)=1$$

In the boson theory, the existence of a massive ground state requires

$$\lambda_b^2+\frac{\lambda_6}{8\pi^2}=4$$

using the mapping between the fermion and the boson theories

O. Aharony, S. Giombi, G. Gur-Ari, J. Maldacena and R. Yacoby

$$\lambda_b=\frac{g}{4\pi}-1\,,\quad \lambda_6=8\pi^2\left(1-\frac{g}{4\pi}\right)^2\left(3-4\frac{g}{g_\sigma}\right)$$

We find that the bosonic and the fermionic conditions are copies of each other

. . . . .and finally an open problem

It would be interesting to explore the implications for the bulk four dimensional AdS dual description of the massive phase.

This is an open problem whose solution is not known at this point. In particular it is unknown whether the bulk theory is just a modification of Vassiliev's theory or whether new fields are required.

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Thanks

The End