Tying together instantons and anti-instantons

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Ginzburg Centennial Conference on Physics, May 30, 2017

Path integral formulation of quantum mechanics

Classical mechanical system $\mathcal{P} \Longrightarrow$ quantum system $(\mathcal{A}, \mathcal{H}, \widehat{H})$

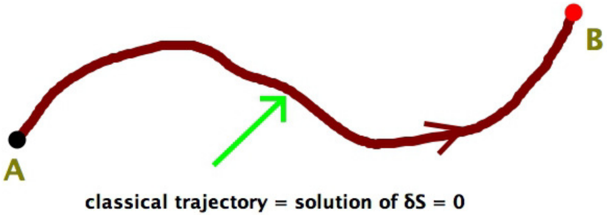
$$\mathcal{A} = \text{algebra of observables}$$

$$\mathcal{H} = \text{space of states}$$

$$\widehat{H} = \text{Hamiltonian}, \qquad \text{generates time evolution}$$

Classical phase space $\mathcal{P}$

$$\omega = d\mathbf{p} \wedge d\mathbf{q}$$



$$\mathbf{S}(\text{trajectory}) = \int_A^B \mathbf{p}\,d\mathbf{q} - H(\mathbf{p},\mathbf{q})\,dt$$

Classical phase space $\mathcal{P}$

Classical equations of motion: Hamilton equations

$$\delta \int_A^B \mathbf{p} d\mathbf{q} - H(\mathbf{p}, \mathbf{q}) dt = 0 \Longrightarrow$$

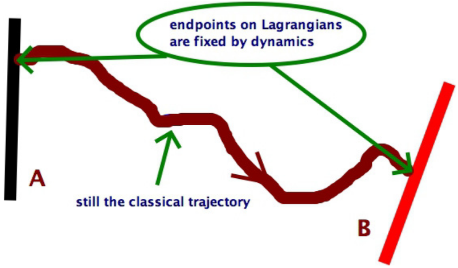
$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$

Classical phase space $\mathcal{P}$, Hamilton equations

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}\,, \qquad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$

! First order equations:
can fix A and B as Lagrangian submanifolds in $\mathcal{P}$

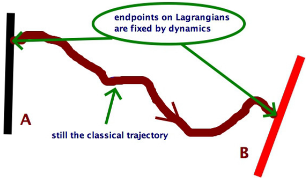


$$\omega|_A = \omega|_B = 0, \qquad \dim(A) = \dim(B) = \frac{1}{2}\dim\mathcal{P}$$

Classical phase space $\mathcal{P}$

Boundary conditions:

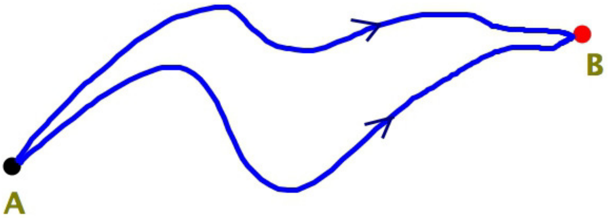
can fix A and B as Lagrangian submanifolds in $\mathcal{P}$



Roughly as points in the configuration space $\mathcal{X}$,
if $\mathcal{P} = T^*\mathcal{X}$

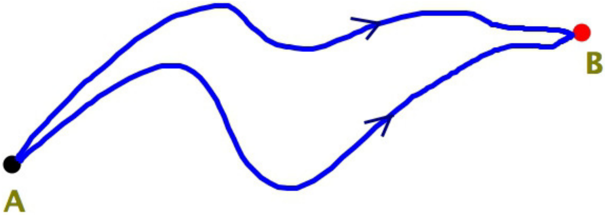
Locally OK, globally interesting and complicated

Quantum picture



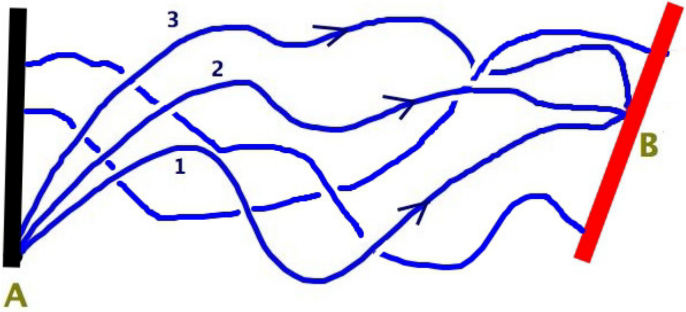
Lots of trajectories

Quantum picture



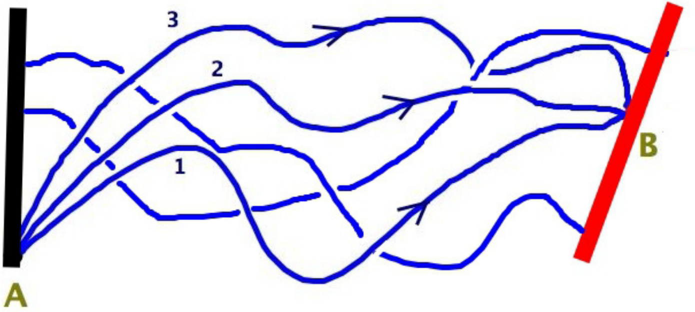
Sum $\sum_i$ over all trajectories $(\mathbf{p}_i, \mathbf{q}_i)(t)$ in the classical phase space $\mathcal{P}$

Quantum picture



$$\langle B|\text{evolution operator}|A\rangle = \sum_{\substack{\text{trajectories}\\ \text{connecting}\\ A \to B}} e^{\frac{iS}{\hbar}}$$

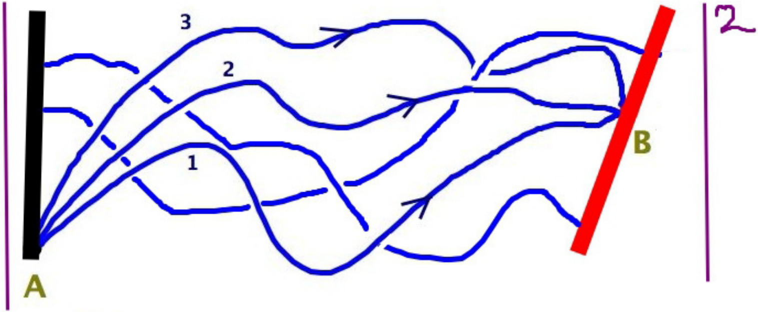
Quantum picture



$$\langle B|\text{evolution operator}|A\rangle = \sum_{\substack{\text{trajectories}\\ \text{connecting}\\ A \to B}} e^{\frac{iS}{\hbar}}$$

amplitude

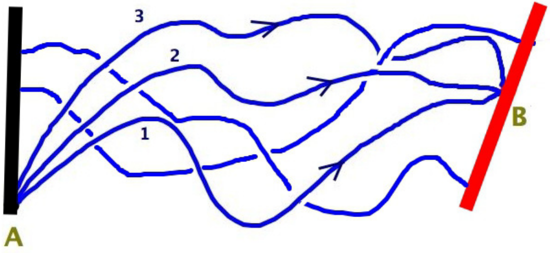
Quantum picture



$$\text{Probability}(A \longrightarrow B) = \left| \sum_{\substack{\text{trajectories} \\ \text{connecting} \\ A \longrightarrow B}} e^{\frac{iS}{\hbar}} \right|^2$$

amplitude

Quantum picture



$$\langle B|\text{evolution operator}|A\rangle = \sum_{\substack{\text{trajectories}\\ \text{connecting}\\ A \to B}} e^{\frac{iS}{\hbar}}$$

amplitude

$$S(\text{trajectory}) = \int_{\text{trajectory}} \mathbf{p}\,d\mathbf{q} - H(\mathbf{p},\mathbf{q})\,dt$$

Quantum picture

Path integral shows that

Evolution operator U_t is a solution of the Schrödinger equation

Quantum picture

Path integral shows that

Evolution operator U_t is a solution of the Schrödinger equation

$$i\hbar \frac{\partial U_t}{\partial t} = \widehat{H} U_t$$

Quantum picture

Path integral shows that

Evolution operator U_t is a solution of the Schrödinger equation

$$i\hbar \frac{\partial U_t}{\partial t} = \widehat{H} U_t \Longrightarrow U_t = \exp\left(-\frac{it}{\hbar}\widehat{H}\right)$$

we assume $\widehat{H}$ is stationary, i.e. no explicit t-dependence

Quantum picture

We want to learn about the spectrum of $\widehat{H}$

$$\widehat{H}|\psi_i\rangle = E_i|\psi_i\rangle$$

$|\psi_i\rangle \in \mathcal{H}$ – complete basis of the space of states

Quantum picture

Path integral helps

to learn about the spectrum of $\widehat{H}$

$$\mathrm{Tr}_{\mathcal{H}}\, U_T = \sum_i e^{-\frac{\mathrm{i}T}{\hbar} E_i}$$

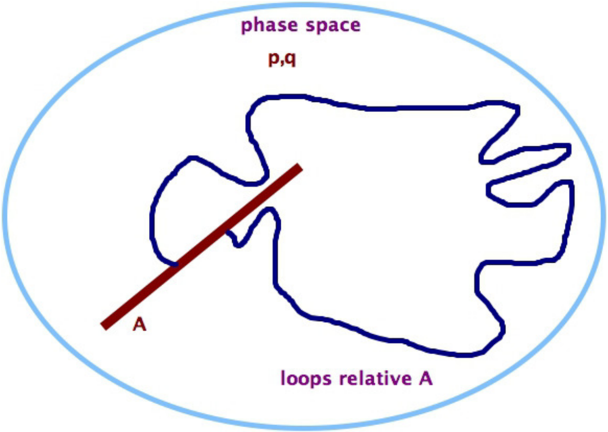
Quantum picture

Path integral helps

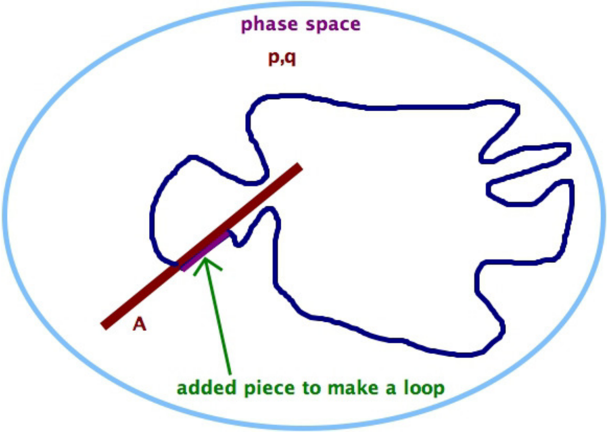
to learn about the spectrum of $\widehat{H}$

$$\sum_i e^{-\frac{\mathrm{i}T}{\hbar}E_i} = \mathrm{Tr}_{\mathcal{H}}\, U_T = \int_{A\in\mathcal{X}} \langle A|U_T|A\rangle$$

$$\mathrm{Tr}_{\mathcal{H}}\ {\color{red}U}_T = \sum_i e^{-\frac{\mathrm{i}T}{\hbar}E_i} = \sum_{A \in \mathcal{X}} \ \sum_{\text{trajectories:}A \to A} e^{\frac{\mathrm{i}S}{\hbar}}$$

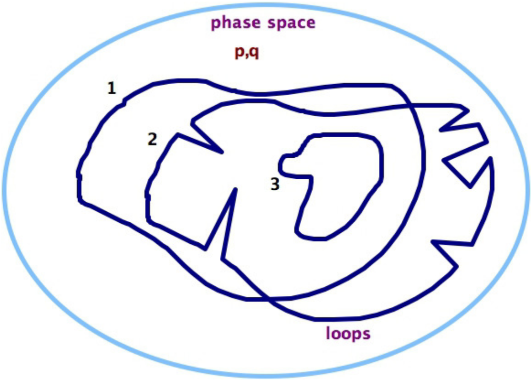


$$\mathrm{Tr}_{\mathcal{H}}\ U_T = \sum_i e^{-\frac{\mathrm{i}T}{\hbar}E_i} = \sum_{A\in\mathcal{X}} \sum_{\text{trajectories}:A\to A} e^{\frac{\mathrm{i}S}{\hbar}}$$



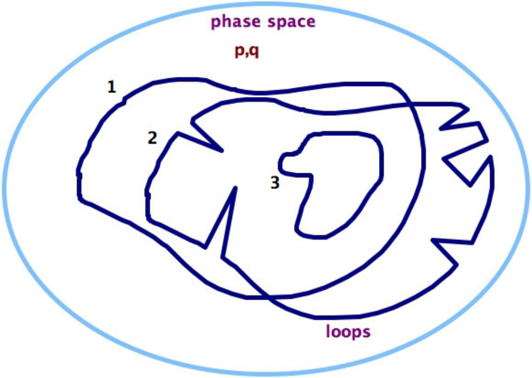
$$\mathrm{Tr}_{\mathcal{H}}\, U_T = \sum_i e^{-\frac{\mathrm{i}T}{\hbar}E_i}$$

$$\sum_{A\in\mathcal{P}} \ \sum_{\mathrm{trajectories}:A\to A} e^{\frac{\mathrm{i}S}{\hbar}}$$



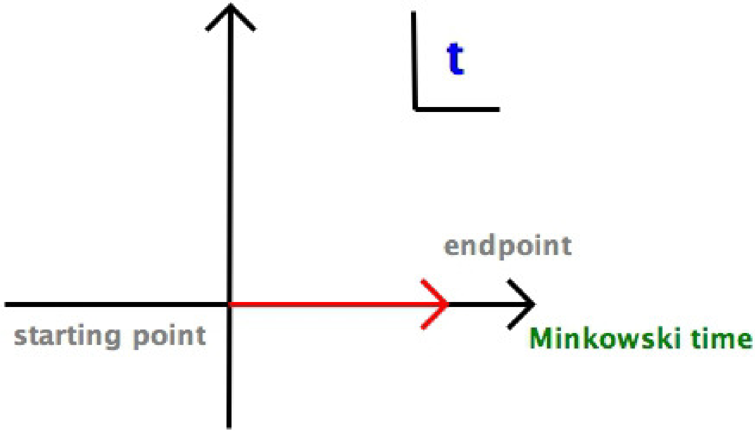
Quantum picture

$$\mathrm{Tr}_{\mathcal{H}}\, U_T = \sum_i e^{-\frac{\mathrm{i}T}{\hbar}E_i}$$

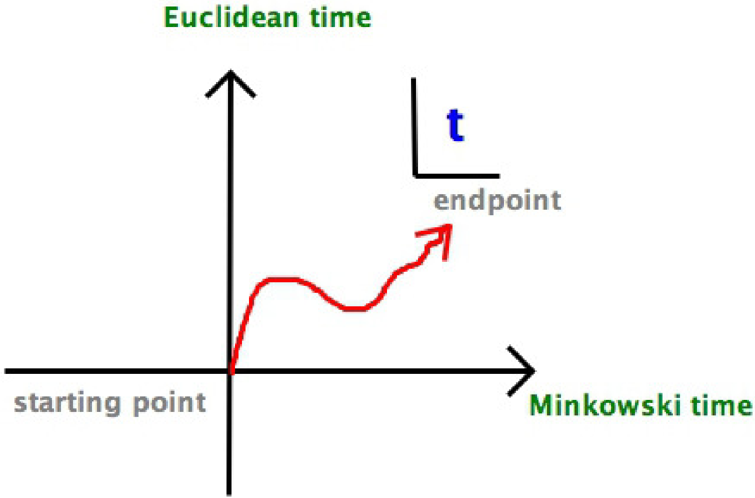


$$= \int_{\mathbf{p}(0)=\mathbf{p}(1),\mathbf{q}(0)=\mathbf{q}(1)} D\mathbf{p}(s)D\mathbf{q}(s)\, \exp \frac{\mathrm{i}}{\hbar}\oint \mathbf{p}d\mathbf{q} - \frac{\mathrm{i}T}{\hbar}\oint_0^1 H(\mathbf{p},\mathbf{q})ds$$

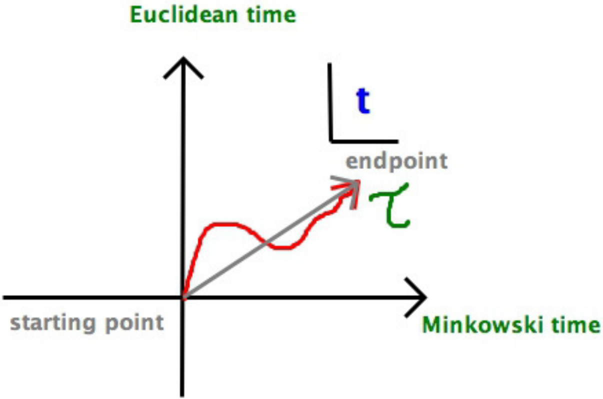
Nature of time



Nature of time

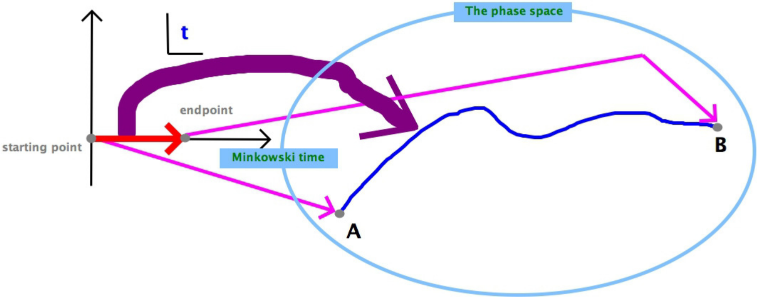


Nature of time

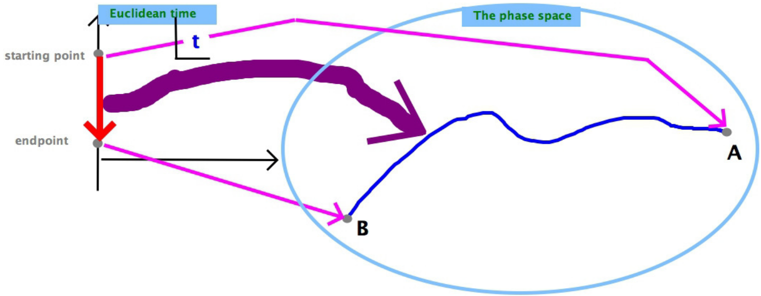


Deform the evolution operator $U_T \mapsto e^{-\frac{\mathrm{i}\tau\widehat{H}}{\hbar}}$

Nature of time



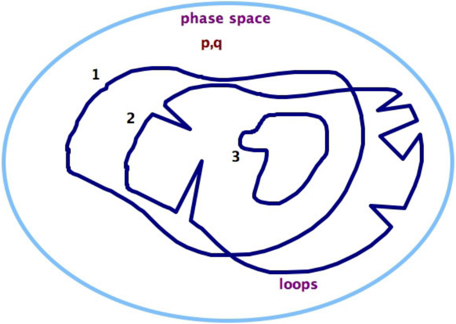
Nature of time: Euclidean arrow of time points south!



So now we compute

$$\mathrm{Tr}_{\mathcal{H}}\, U^{E}_{T} = \sum_i e^{-\frac{T}{\hbar}E_i}$$

$$\sum_{A\in\mathcal{P}} \sum_{\text{trajectories}:A\to A} \exp \frac{\mathrm{i}\oint \mathbf{p}d\mathbf{q} - T\oint H(\mathbf{p}(s),\mathbf{q}(s))ds}{\hbar}$$

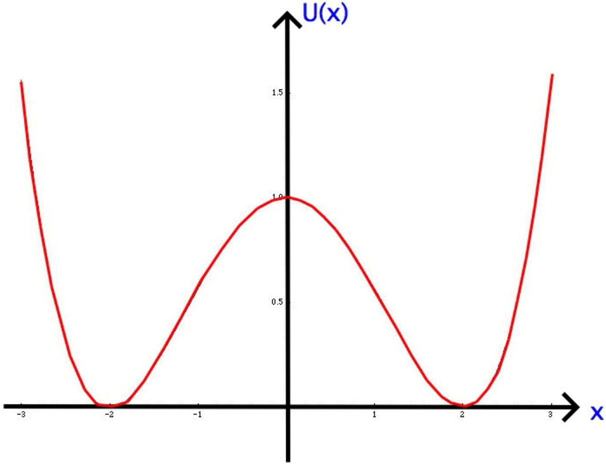


Same loops, different action

A textbook problem

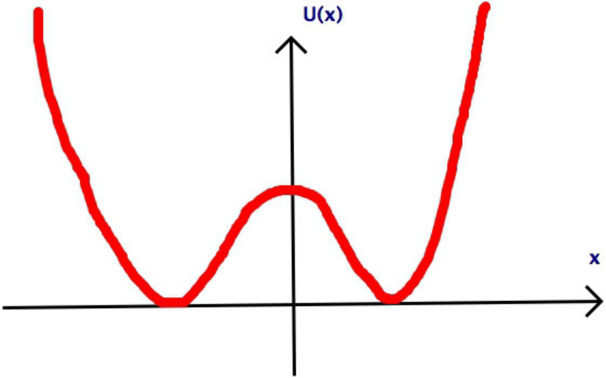
Level splitting

Double well potential with symmetry $x \to -x$



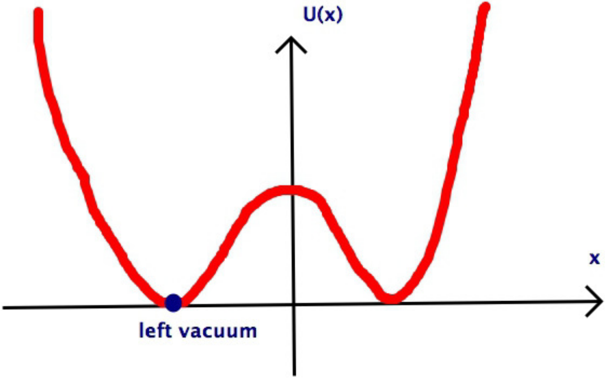
$$H(p,x) = \frac{p^2}{2} + U(x)$$

Double well potential



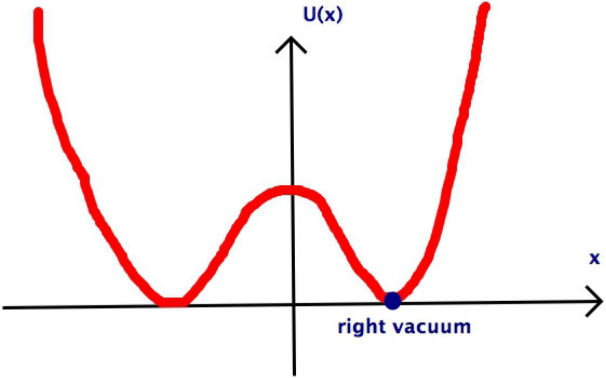
$$U(x) = \frac{\lambda}{4}\left(x^2 - x_0^2\right)^2$$

Double well potential with symmetry $x \to -x$



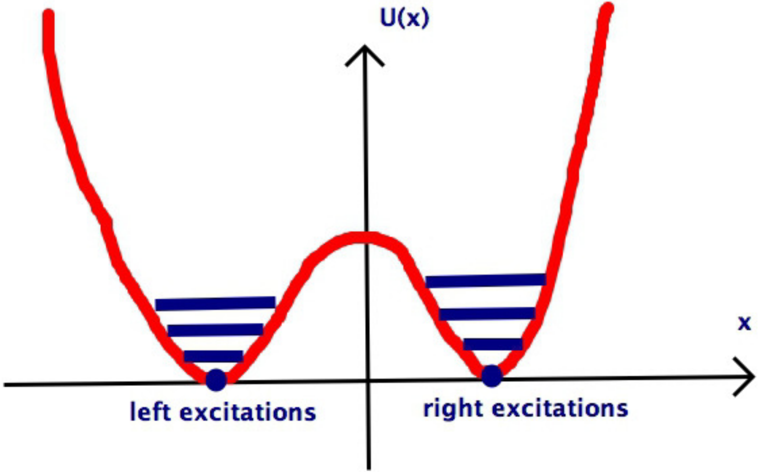
$$p = 0, x = -x_0$$

Double well potential with symmetry $x \to -x$

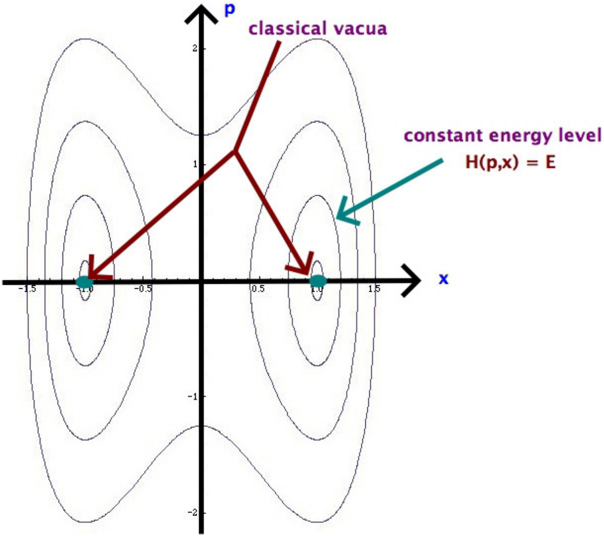


$$p = 0, x = +x_0$$

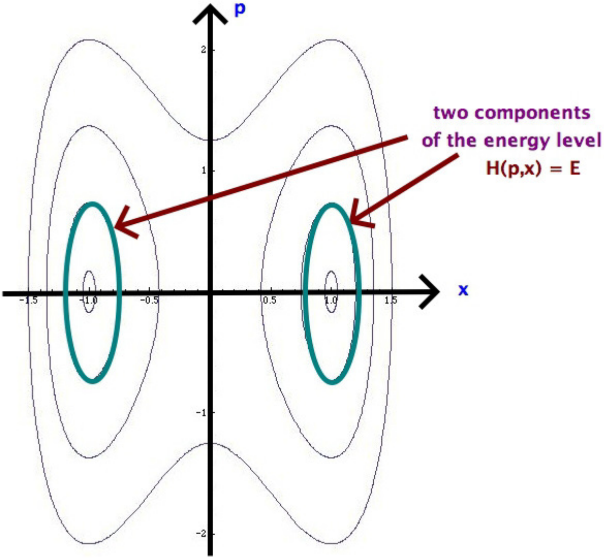
Double well potential with symmetry $x \to -x$



Classical energy levels

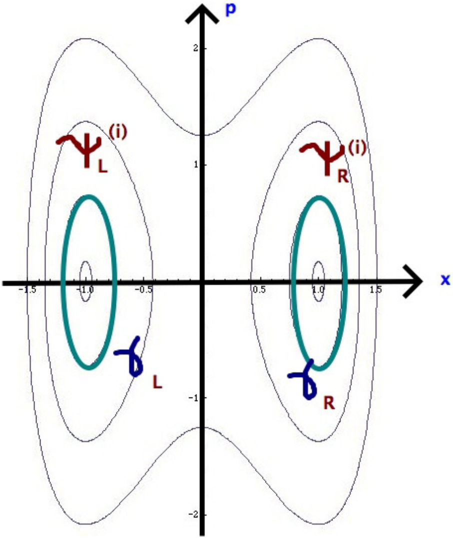


Classical energy levels



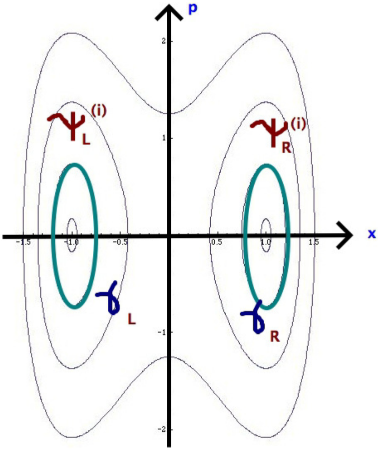
Classical life is doubly degenerate

From classical to quantum energy levels



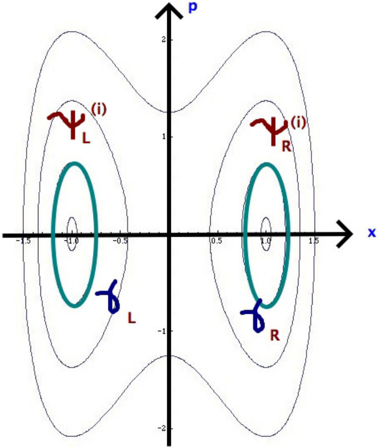
Excitations correspond to the Bohr-Sommerfield orbits

From classical to quantum energy levels



Bohr-Sommerfield orbits: $\oint_{\gamma_{L,R}} p dx = 2\pi\hbar N_i, \; N_i \in \mathbb{Z} + \ldots$

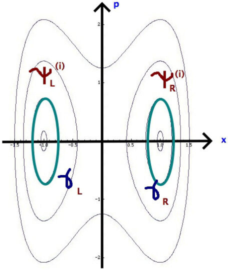
From classical to quantum energy levels



Bohr-Sommerfield orbits: $\oint_{\gamma_{L,R}} p dx = 2\pi\hbar N_i$, $N_i \in \mathbb{Z} + \ldots$

The spectrum **is** doubly degenerate to all orders in $\hbar$ expansion

From classical to quantum energy levels



$$\Psi_{\pm}^{(i)} = \frac{1}{\sqrt{2}}\left(\Psi_L^{(i)} \pm \Psi_R^{(i)}\right)$$

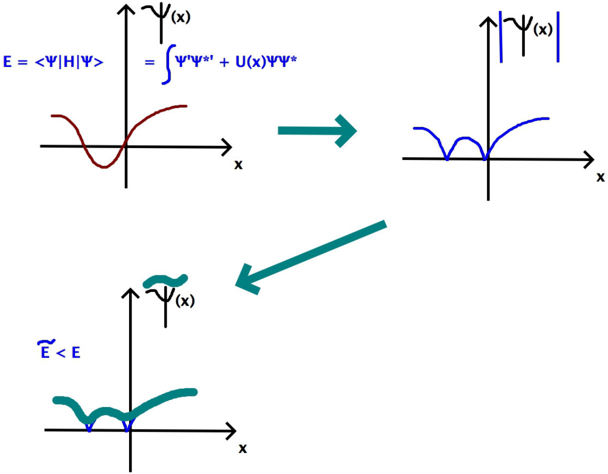
The spectrum **is** doubly degenerate to all orders in $\hbar$ expansion

$$E_+^{(i)} - E_-^{(i)} = O(\hbar^\infty)$$

Quantum energy levels

The spectrum **cannot be** doubly degenerate,
certainly not the ground state,
as Feynman's variational method quickly shows

Feynman's variational method quickly shows



Here, $\Psi(x) \propto \psi_2(x_0)\psi_1(x) - \psi_1(x_0)\psi_2(x)$

The textbook solution

Textbook solution: compute

$$\mathrm{Lim}_{T\to+\infty}\langle -x_0|\ U_T^E\ |x_0\rangle \approx e^{-TE_+^{(0)}}|\Psi_+^{(0)}(x_0)|^2 - e^{-TE_-^{(0)}}|\Psi_-^{(0)}(x_0)|^2$$

$$\mathrm{Lim}_{T\to+\infty}\langle x_0|\ U_T^E\ |x_0\rangle \approx e^{-TE_+^{(0)}}|\Psi_+^{(0)}(x_0)|^2 + e^{-TE_-^{(0)}}|\Psi_-^{(0)}(x_0)|^2$$

$$\Psi_\pm(x) = \pm\Psi_\pm(-x)$$

Textbook solution: compute for $T \to \infty$

$$\langle -x_0 |\ U_T^E\ | x_0 \rangle = \int_{\text{paths}: x_0 \to (-x_0)} DpDx\, e^{\frac{\mathrm{i} \int p dx - \int_0^T H(p,x) dt}{\hbar}}$$

$$\langle x_0 |\ U_T^E\ | x_0 \rangle = \int_{\text{paths}: x_0 \to x_0} DpDx\, e^{\frac{\mathrm{i} \int p dx - \int_0^T H(p,x) dt}{\hbar}}$$

Textbook solution: compute for small $\hbar \to 0$, $T \to \infty$

$$\langle -x_0 | \; U_T^E \; | x_0 \rangle = \int_{\text{paths}: x_0 \to (-x_0)} DpDx \; e^{\frac{\mathrm{i} \int p dx - \int_0^T H(p,x) dt}{\hbar}}$$

$$\langle x_0 | \; U_T^E \; | x_0 \rangle = \int_{\text{paths}: x_0 \to x_0} DpDx \; e^{\frac{\mathrm{i} \int p dx - \int_0^T H(p,x) dt}{\hbar}}$$

Textbook solution: $\hbar \to 0 \Longrightarrow$ **saddle points** for $T \to \infty$

$$\delta \left(\mathrm{i} \int p dx - \int_0^T H(p,x) dt \right) = 0$$

$$\mathrm{i}\dot{x} = \frac{\partial H}{\partial p}$$

$$-\mathrm{i}\dot{p} = \frac{\partial H}{\partial x}$$

Textbook solution: $\hbar \to 0 \Longrightarrow$ **saddle points** for $T \to \infty$

$$\delta \left(\mathrm{i} \int p dx - \int_{-T/2}^{T/2} H(p,x) dt \right) = 0$$

Hamilton equations with a twist, by 90 degrees

$$\mathrm{i}\dot{x} = \frac{\partial H}{\partial p} = p$$

$$-\mathrm{i}\dot{p} = \frac{\partial H}{\partial x} = U'(x)$$

$$x(-T/2) = x_0, \qquad x(T/2) = \pm x_0$$

Textbook solution: $\hbar \to 0 \Longrightarrow$ **saddle points** for $T \to \infty$

$$\delta\left(\mathrm{i}\int p dx - \int_{-T/2}^{T/2} H(p,x) dt\right) = 0$$

Hamilton equations with a twist, by 90 degrees

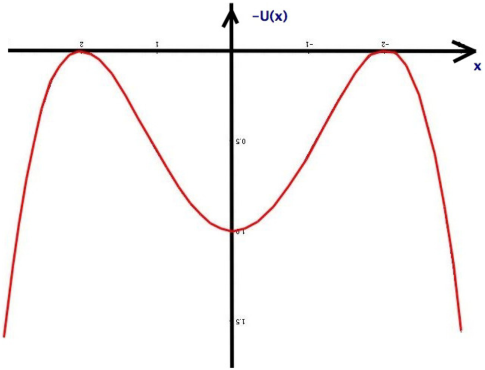
$$\mathrm{i}\dot{x} = p\,, \qquad -\mathrm{i}\dot{p} = U'(x) \Longrightarrow H(p,x) = const$$

$$x(-T/2) = x_0, \qquad x(T/2) = \pm x_0$$

Textbook solution: $\hbar \to 0 \Longrightarrow$ **saddle points** for $T \to \infty$

Textbooks usually solve for p, and get

$$\ddot{x} = U'(x) \Longrightarrow -\frac{1}{2}(\dot{x})^2 + U(x) = const$$

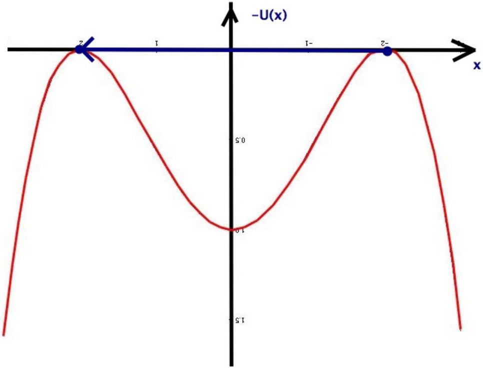


Finite action saddle point for $T = \infty$

$$''Energy'' = -\frac{1}{2}(\dot{x})^2 + U(x) = 0$$

Instanton: $\dot{x} + \sqrt{2U(x)} = 0$

$$x(-\infty) = x_0, \qquad x(+\infty) = -x_0$$

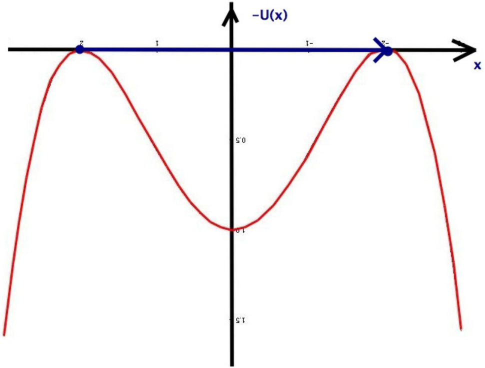


Finite action saddle point for $T = \infty$

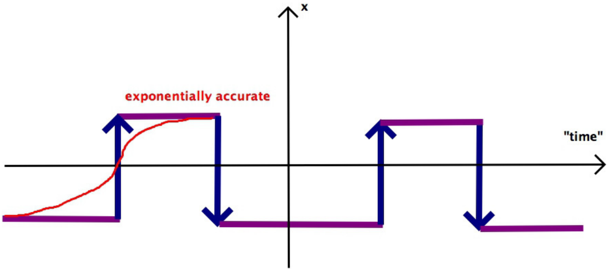
$$"\,Energy" = -\frac{1}{2}(\dot{x})^2 + U(x) = 0$$

Anti-instanton: $\dot{x} - \sqrt{2U(x)} = 0$

$$x(-\infty) = -x_0, \qquad x(+\infty) = +x_0$$



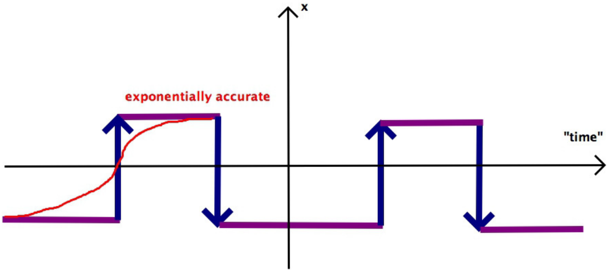
And then the textbooks close in fast:
Superpose Instantons and Anti-Instantons



+ some reasonable estimates
of the effects of fluctuations one arrives at

$$E_{+}^{(0)} - E_{-}^{(0)} \propto e^{-2S_{\mathrm{i}}/\hbar}$$

Superpose Instantons and Anti-Instantons

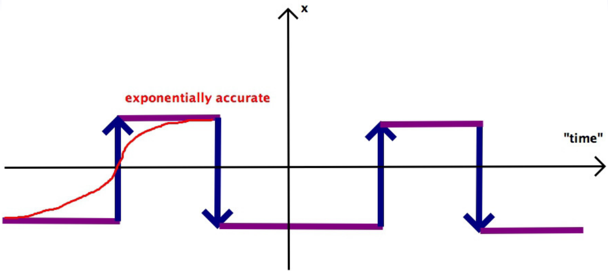


$+$ some reasonable estimates
of the effects of fluctuations one arrives at

$$E_+^{(0)} - E_-^{(0)} \propto e^{-2S_{\rm i}/\hbar}$$

$$S_{\rm i} = \int_{-x_0}^{x_0} \sqrt{2U(x)}dx\,, \qquad \textit{an instanton action}$$

Superpose Instantons and Anti-Instantons



$+$ some reasonable estimates
of the effects of fluctuations one arrives at

$$E_{+}^{(0)} - E_{-}^{(0)} = e^{-2S_{\mathbf{i}}/\hbar}\,(1 + \ldots)$$

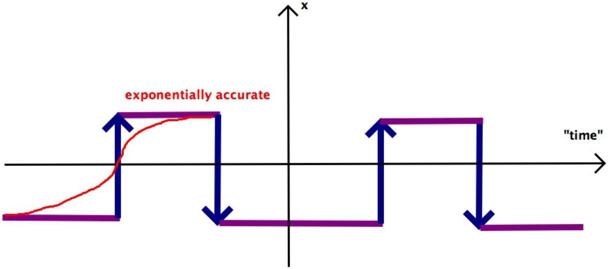
loop expansion

$$S_{\mathbf{i}} = \int_{-x_0}^{x_0} \sqrt{2U(x)}\,dx\,, \qquad \textit{an instanton action}$$

With all due admiration to the authors of this method

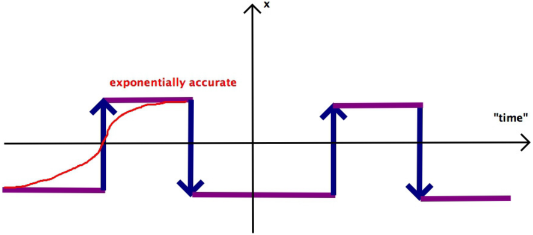
A. Polyakov, S. Coleman, . . .

I have always been a little bit worried:



The $\mathcal{I} - \bar{\mathcal{I}}$ superposition is not a saddle point!

Superposition of Instantons and Anti-Instantons aka the instanton gas

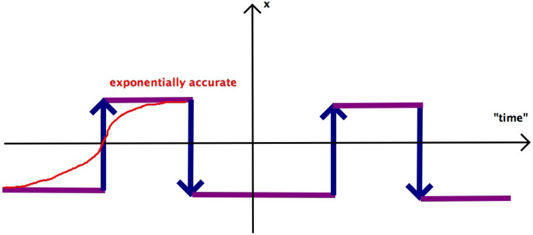


is not a saddle point! Fluctuations contain tadpoles: $\delta S \neq 0$

Interpretation: tadpoles move us toward the true saddle points

A. Schwarz: "Newton's method" (E. Bogomolny'80)

Superposition of Instantons and Anti-Instantons aka the instanton gas



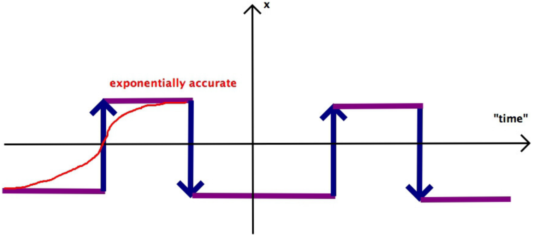
is not a saddle point! Fluctuations contain tadpoles: $\delta S \neq 0$

E. Bogomolny (1980) has improved this method: tadpoles as sources

$$S \to S - \frac{1}{2}\delta S \left(\delta^2 S\right)^{-1} \delta S$$

$\Longrightarrow$ interaction potential of interaction between the $\mathcal{I}$ and $\bar{\mathcal{I}}$

Non-ideal instanton gas



is not a saddle point! Fluctuations contain tadpoles: $\delta S \neq 0$

But where are the true saddle points?

$$S \to S - \frac{1}{2}\delta S \left(\delta^2 S\right)^{-1} \delta S - \dots$$

$$\mathcal{I}\bar{\mathcal{I}} \to \mathcal{I}\bar{\mathcal{I}} - \left(\delta^2 S\right)^{-1} \delta S - \dots \to ????$$

Change gears for a bit

Back to path integral

$$Z = \int_{\mathcal{F}ields} [D\phi]\ e^{-\frac{S(\phi)}{\hbar}}$$

Topological renormalisation group

Well-known general idea: view the path integral

$$Z = \int_{\mathcal{F}} [D\phi] \; e^{-\frac{S(\phi)}{\hbar}}$$

as a period:

$$Z = \int_{\Gamma} \Omega_{\hbar}, \qquad \Omega_{\hbar} = [D\phi] \; e^{-\frac{S(\phi)}{\hbar}}$$

a middle-dimensional contour $\Gamma \subset \mathcal{F}^{\mathbb{C}}$

Topological renormalisation group

The period does not change when the contour is deformed

$$Z = \int_\Gamma \Omega_\hbar, \qquad \Omega_\hbar = [D\phi]\ e^{-\frac{S(\phi)}{\hbar}}$$

Optimal choice of the contour:
gradient flow for some hermitian metric h on $\mathcal{F}^{\mathbb{C}}$

$$V = \nabla^h (\mathrm{Re}(S/\hbar))$$

Topological renormalisation group

The period does not change when the contour is deformed

$$Z = \int_{\Gamma} \Omega_{\hbar}, \qquad \Omega_{\hbar} = [D\phi] \; e^{-\frac{S(\phi)}{\hbar}}$$

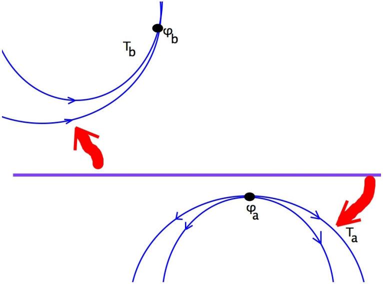
gradient flow for some hermitian metric h on $\mathcal{F}^{\mathbb{C}}$

$$V = \nabla^{h} \left(\mathrm{Re}(S/\hbar) \right)$$

$$\Gamma_0 = \mathcal{F} \longrightarrow \Gamma_t = e^{tV} \left(\mathcal{F} \right)$$

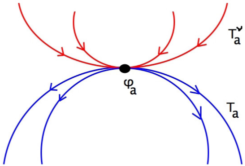
Fixed points
of the topological renormalisation group

$$\Gamma_t \longrightarrow \Gamma_\infty \sim \sum_{\mathbf{a}} n_{\mathbf{a}} \; T_{\mathbf{a}}$$



Complex saddle points for partition functions

$$Z = \sum_{\mathbf{a}} n_{\mathbf{a}} \int_{T_{\mathbf{a}}} \Omega_{\hbar},$$



$T_{\mathbf{a}}$ - Lefschetz thimbles (F. Pham'83)
emanating from the critical point $\varphi_{\mathbf{a}}$
$dS|_{\varphi_{\mathbf{a}}} = 0$

Complex saddle points for partition functions

$$Z = \sum_{\mathbf{a}} n_{\mathbf{a}} \int_{T_{\mathbf{a}}} \Omega_{\hbar},$$

$T_{\mathbf{a}}$ - Lefschetz thimbles
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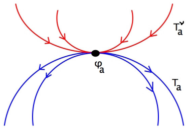


A. Varchenko, A. Givental'82
F. Pham'83
V. Arnol'd-A. Varchenko-S. Gusein-Zade'83
S. Cecotti'91
S. Cecotti, C. Vafa'91
A. Losev, NN'93
A. Iqbal, K. Hori, C. Vafa'00
E. Witten'09

Path integral as period

The action in $e^{-S/\hbar}$

$$S = -\mathrm{i} \int_\gamma \mathbf{p} d\mathbf{q} + \int_0^1 ds\, H(\mathbf{p}(s), \mathbf{q}(s))$$

The fields: $\mathcal{F} = L\mathcal{P}$

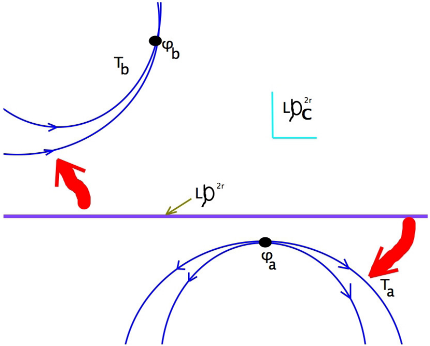
is the space of parametrized loops $\varphi : S^1 \to \mathcal{P}$

$$\varphi(s) = (\,\mathbf{p}(s), \mathbf{q}(s)\,) \in \mathcal{P}, \quad \varphi(s+1) = \varphi(s)\ .$$

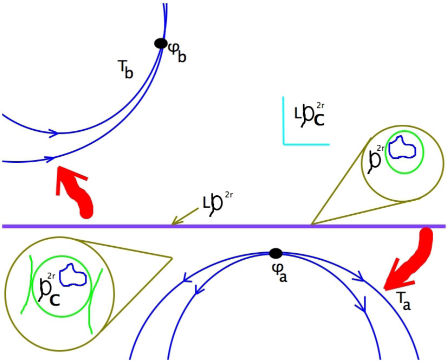
Complexify the classical picture

- Complex phase space $(\mathcal{P}_\mathbb{C}, \varpi_\mathbb{C})$, $\varpi_\mathbb{C} = d\mathbf{p}_\mathbb{C} \wedge d\mathbf{q}_\mathbb{C}$
- Holomorphic Darboux coordinates $(\mathbf{p}_\mathbb{C}, \mathbf{q}_\mathbb{C})$

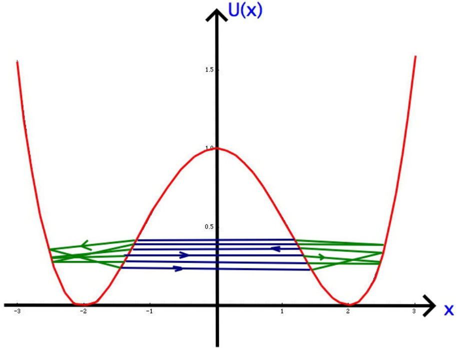
Now contour is in the complexified loop space



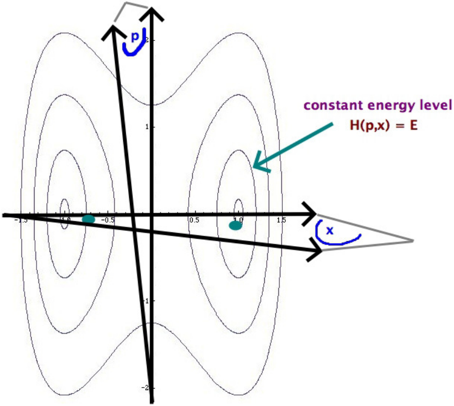
Contour in the complexified loop space



Complex Saddle Points: qualitative picture

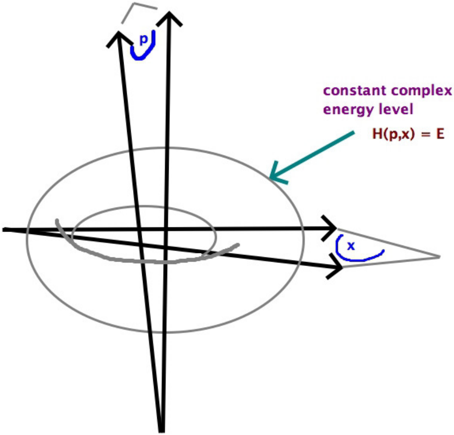


Complex Saddle Points: qualitative picture



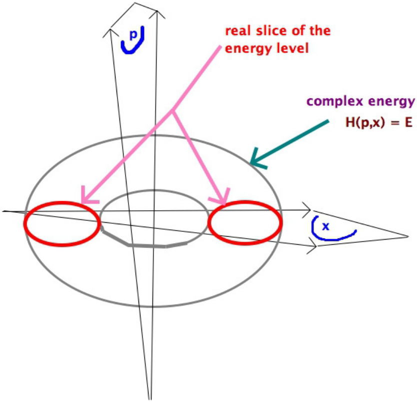
The complexified phase space is $\mathbb{C}^2 \approx \mathbb{R}^4$ now

Complex Saddle Points: qualitative picture



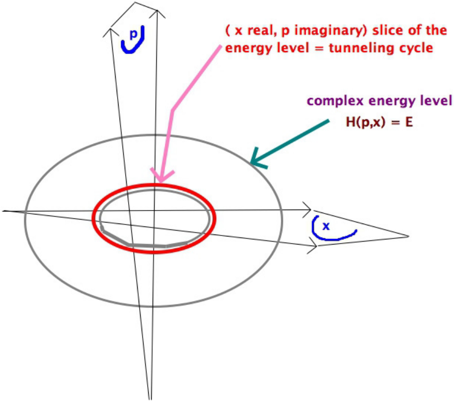
The complexified energy level space is now an elliptic curve $\mathcal{E} \approx \mathbb{T}^2$

Complex Saddle Points: qualitative picture

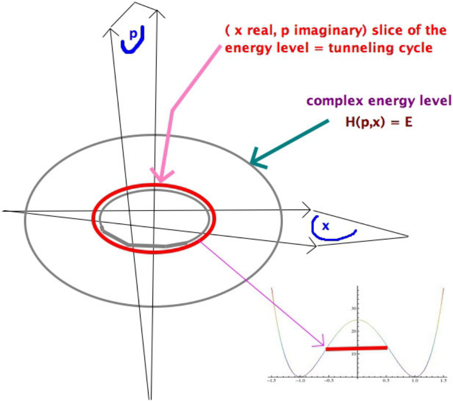


Our old friends real energy levels are the real slices of that $\mathbb{T}^2$

Complex Saddle Points: qualitative picture

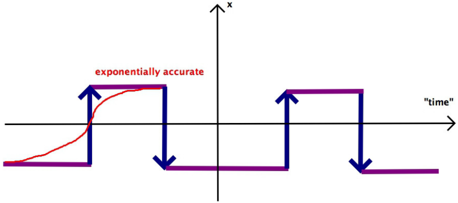


Complex Saddle Points: qualitative picture



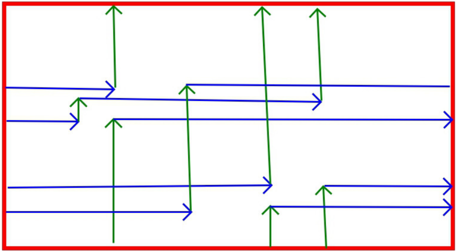
Complex Saddle Points: qualitative picture

Instanton gas



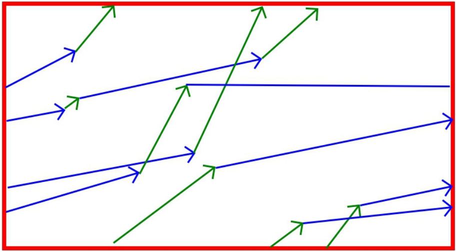
Maps to piecewise linear paths on the torus:

Torus cycles: winding $(3,4)$



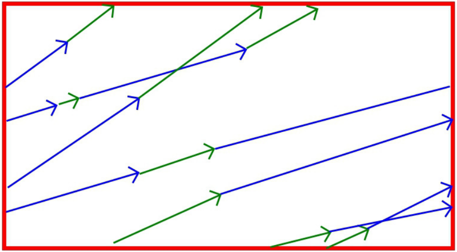
This $\uparrow\uparrow\uparrow$ is not a critical point!

Torus cycles: winding $(3, 4)$



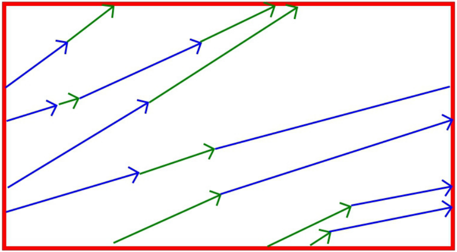
The gradient flow moves $\uparrow\uparrow\uparrow$ towards a critical point!

Torus cycles: winding $(3,4)$



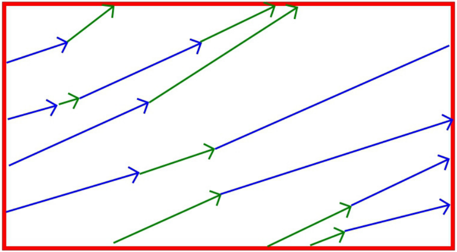
It moves ...

Torus cycles: winding $(3, 4)$



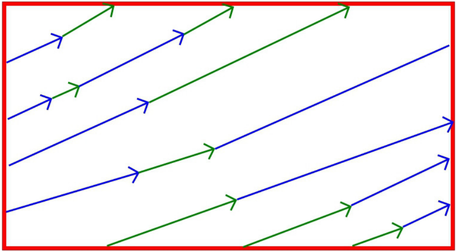
And moves . . .

Torus cycles: winding $(3, 4)$



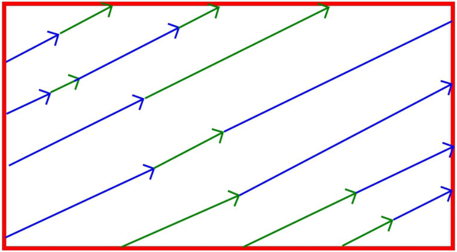
And moves . . .

Torus cycles: winding $(3, 4)$



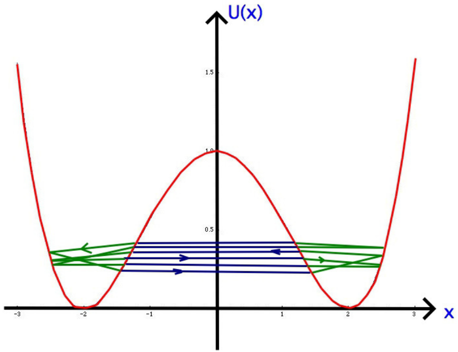
And moves further down ...

Torus cycles: winding $(3, 4)$

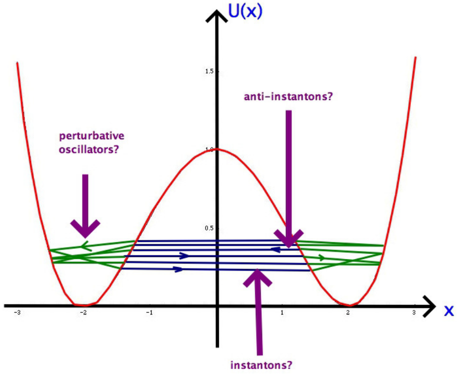


Until we reach the critical point

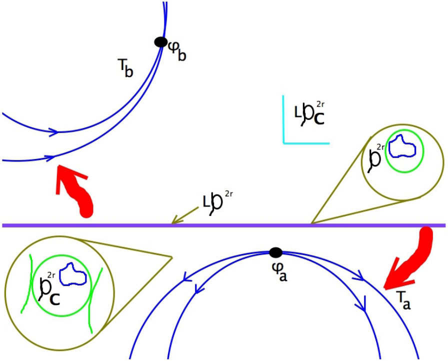
Where are the instantons?



Where are the instantons and anti-instantons?



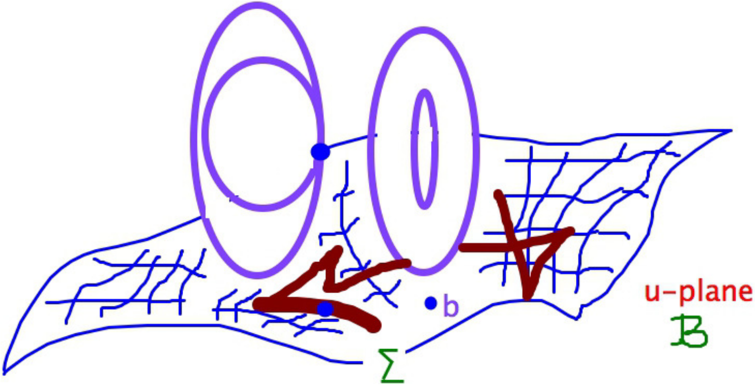
What are the critical points φ_a's in general?



With an additional assumption

of "algebraic integrability"

$\mathcal{P}_{\mathbb{C}}$ fibers over $\mathcal{B}_{\mathbb{C}} \subset \mathbb{C}^r$



The complex critical points are :

rational windings on tori

$\mathbb{T}^{2r}$ - complex tori (abelian varieties)

Two winding vectors

$\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$

Algebraic integrability :
action variables

$$a^i = \oint_{A_i} \mathbf{p} d\mathbf{q}\,, \qquad a_{D,i} = \oint_{B^i} \mathbf{p} d\mathbf{q}$$

$2r$ variables on r-dimensional space: non-independent

$$\mathbf{a}_D d\mathbf{a} = d\mathcal{F}$$

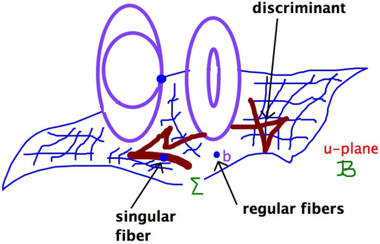
$\mathcal{F}$-prepotential of the effective low-energy $\mathcal{N} = 2$ action

Algebraic integrability :

action variables

$$a^i = \oint_{A_i} \mathbf{p} d\mathbf{q} , \qquad a_{D,i} = \oint_{B_i} \mathbf{p} d\mathbf{q}$$

Well-defined on $\widetilde{\mathcal{B}_{\mathbb{C}} \setminus \Sigma}$



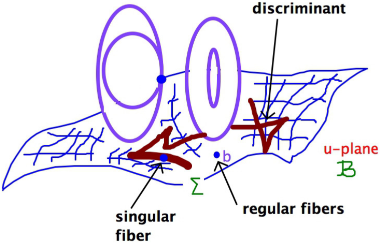
Monodromy in $Sp(2r, \mathbb{Z})$

Algebraic integrability :

action variables near degeneration locus Σ

Complex codimension 1 stratum: one vanishing cycle

$$a \to 0, \qquad a_D = \frac{1}{2\pi i} a \log(a) + \ldots$$



Algebraic integrability :

Feature of complex angle variables:

Double periodicity

$$\oint_{A_i} \varpi_j = \delta^i_j \,, \qquad \oint_{B^i} \varpi_j = \tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}$$

$$\phi_i \sim \phi_i + n_i + \sum_{j=1}^{r} \tau_{ij} m^j \,, \qquad n_i, m^k \in \mathbb{Z}$$

Now we can solve for the Complex Saddle Points

$$\delta S = 0 \quad \Leftrightarrow$$

$$\mathrm{i}\frac{d\mathbf{p}}{ds} = -\frac{\partial H}{\partial \mathbf{q}}, \qquad \mathrm{i}\frac{d\mathbf{q}}{ds} = \frac{\partial H}{\partial \mathbf{p}}$$

Now we can solve for the Complex Saddle Points

$$\delta S = 0 \quad \Leftrightarrow$$

$$\mathrm{i}\frac{d\mathbf{p}}{ds} = -\frac{\partial H}{\partial \mathbf{q}}, \qquad \mathrm{i}\frac{d\mathbf{q}}{ds} = \frac{\partial H}{\partial \mathbf{p}}$$

$$\Longrightarrow$$

the critical loop $\varphi_{\mathbf{a}} = [\gamma(s)]$ sits in a particular fiber $\mathbb{T}^{2r}_b$, $b \in \mathcal{B}_{\mathbb{C}}$

Complex Saddle Points

Pass to action-angle variables

$$\frac{d\phi}{ds} = \mathrm{i}\frac{\partial H}{\partial \mathbf{a}}, \qquad \frac{d\mathbf{a}}{ds} = 0$$

$$\Longrightarrow$$

the critical loop $\varphi_{\mathbf{a}} = [\gamma(s)]$ sits in a particular fiber $\mathbb{T}^{2r}_{b}$, $b \in \mathcal{B}_{\mathbb{C}}$
where the motion is a straight line in the angle variables

$$\phi(s) = \phi(0) + \mathbf{\Omega}\, s$$

$$\mathbf{\Omega} = \mathrm{i}\frac{\partial H}{\partial \mathbf{a}}$$

Pass to action-angle variables

$$\frac{d\phi}{ds} = \mathrm{i}\frac{\partial H}{\partial \mathbf{a}}, \qquad \frac{d\mathbf{a}}{ds} = 0$$

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$$\phi(s) = \phi(0) + \mathbf{\Omega}\, s$$

$$\mathbf{\Omega} = \mathrm{i}\frac{\partial H}{\partial \mathbf{a}}$$

The fiber b is fixed by $\phi(0) = \phi(1)$ up to the periods

$$\phi(1) = \phi(0) + \mathbf{n} + \tau \cdot \mathbf{m}$$

Superpotential for Complex Saddle Points

$$\mathbf{\Omega} = \mathbf{n} + \tau \cdot \mathbf{m} = \mathrm{i}\frac{\partial H}{\partial \mathbf{a}}$$

for some integer vectors $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$

$$\Leftrightarrow d\mathcal{W}_{\mathbf{n},\mathbf{m}} = 0$$

$$\mathcal{W}_{\mathbf{n},\mathbf{m}}(b) = \mathbf{n} \cdot \mathbf{a}(b) + \mathbf{m} \cdot \mathbf{a}_D(b) - H(b)$$

Well-defined on $\widehat{\mathcal{B}_{\mathbb{C}} \backslash \Sigma}$

Landau-Ginzburg description!

for integer vectors $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$

$$d\mathcal{W}_{\mathbf{n},\mathbf{m}} = 0$$

$$\mathcal{W}_{\mathbf{n},\mathbf{m}}(b) = \mathbf{n} \cdot \mathbf{a}(b) + \mathbf{m} \cdot \mathbf{a}_D(b) - H(b)$$

Supersymmetric $d = 2$ $\mathcal{N} = 2$ LG model

So, now we are facing the next question :

Where are the critical points of the superpotential $\mathcal{W}_{\mathbf{n},\mathbf{m}}$?

Picard-Lefschetz theory :

In the limit where $T \to \infty$ degeneration $b \to b_*$

$$b_* \in \Sigma$$

$codim_{\mathbb{C}} = 1$ stratum: one vanishing cycle

$$a \sim T_0(b - b_*) \to 0, \qquad a_D \sim 2S_{\mathbf{i}} + \frac{1}{2\pi \mathrm{i}} a \left(\log(a) - 1\right) + \dots$$

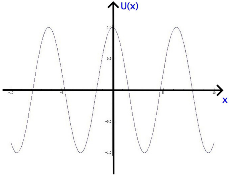
$$\frac{\partial a}{\partial b} \to T_0, \qquad \frac{\partial a_D}{\partial b} \sim \frac{T_0}{2\pi \mathrm{i}} \log\left(T_0(b - b_*)\right) + \dots$$

can make estimates ...

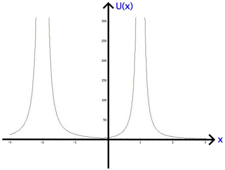
Algebraic integrability

$r = 1$, one degree of freedom, examples

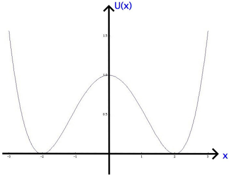
$$\varpi = dp \wedge dx, \qquad H = \frac{1}{2}p^2 + U(x)$$



Mathieu,



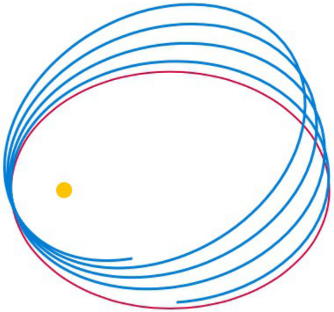
Heun,



Higgs

Another curious quantum-mechanical example

Probe particle in a black hole background



Another curious quantum-mechanical example

Probe particle in a mass M Schwarzschild black hole background

Fixed energy E, fixed angular momentum $L \Longrightarrow$

elliptic curve in the complexified phase space

$$\left(\frac{L}{r^2}\frac{dr}{d\varphi}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

Another curious quantum-mechanical example

Probe particle in a mass M Schwarzschild black hole background

Fixed energy E, fixed angular momentum $L \Longrightarrow$

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$$\left(\frac{L}{r^2}\frac{dr}{d\varphi}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

$$p^2 = E^2 - (1 - 2Mz)\left(1 + z^2\right) , \qquad d\varphi = L\frac{dz}{p}$$

In the limit $T \to \infty$

the elliptic curve (the energy level) degenerates

the action variables near degeneration locus Σ

$$a \sim T_0(b - b_*) \to 0, \qquad a_D \sim 2S_{\mathbf{i}} + \frac{1}{2\pi \mathrm{i}} a \left(\log(a) - 1\right) + \dots$$

$$\mathrm{i}\tau = m\frac{\partial a}{\partial b} + n\frac{\partial a_D}{\partial b} \sim mT_0 + \frac{nT_0}{2\pi \mathrm{i}} \log\left(\frac{b - b_*}{b_0}\right) + \dots$$

In the limit $T \to \infty$

the complex energy is thus fixed to be

$$E(b) \sim b_{m,n} = b_* + b_0 e^{-\frac{2\pi i m}{n}} e^{-\frac{2\pi T}{n T_0}} ,$$

Two quantum numbers!

$n = 1, 2, \ldots$, and $m = 0, 1, \ldots, n-1$

In the limit $T \to \infty$

$$E(b) \sim b_{m,n} = b_* + b_0 e^{-\frac{2\pi i m}{n}} e^{-\frac{2\pi T}{n T_0}},$$

Two quantum numbers!

$n = 1, 2, \ldots,$
and $m = 0, 1, \ldots, n-1$

For $(m, n) = (0, 1)$ these are BI-ons of G.Dunne and M.Unsal'13-15
Also, G.Basar, R.Dabrowski, G.Dunne, M.Shifman, M.Unsal, ...

In the limit $T \to \infty$

$$E(b) \sim b_{m,n} = b_* + b_0 e^{-\frac{2\pi i m}{n}} e^{-\frac{2\pi T}{n T_0}} \,,$$

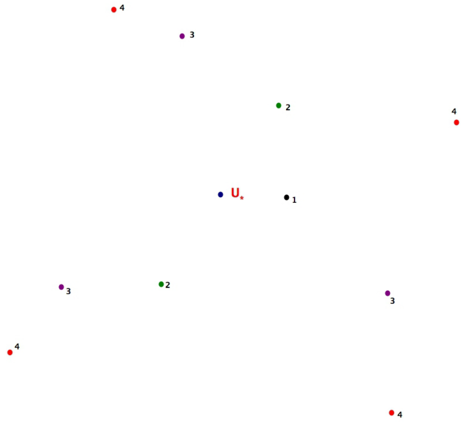
Two quantum numbers: emergent topology!

$n = 1, 2, \ldots$, and $m = 0, 1, \ldots, n-1$

For $(m, n) = (0, 1)$ these are BI-ons of G.Dunne and M.Unsal'13-15
Also, G.Dunne,R.Dabrowski, G.Basar, M.Unsal, M.Shifman, ...

Complex energy In the limit $T \to \infty$

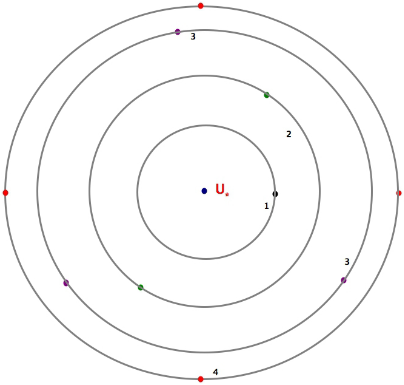
$$E(b) \sim b_{m,n} = b_* + b_0 e^{-\frac{2\pi i m}{n}} e^{-\frac{2\pi T}{n T_0}} ,$$



$$n \in \mathbb{Z}_+, \; 0 \leq m < n$$

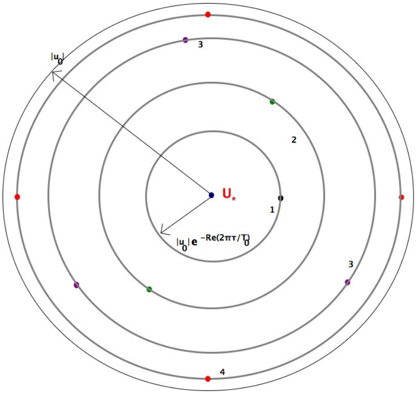
Complex energy In the limit $T \to \infty$

$$E(b) \sim b_{m,n} = b_* + b_0 e^{-\frac{2\pi i m}{n}} e^{-\frac{2\pi T}{n T_0}}$$

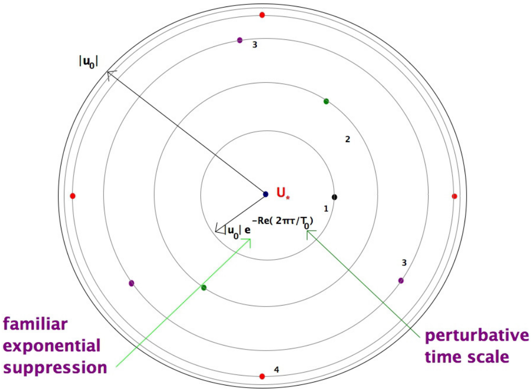


Complex energy In the limit $T \to \infty$

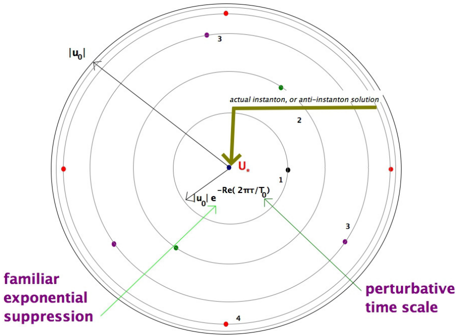
$$E(b) \sim b_{m,n} = b_* + b_0 e^{-\frac{2\pi i m}{n}} e^{-\frac{2\pi T}{n T_0}}$$



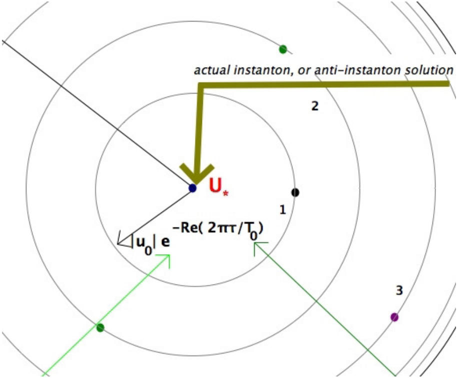
Fine structure of the saddle points



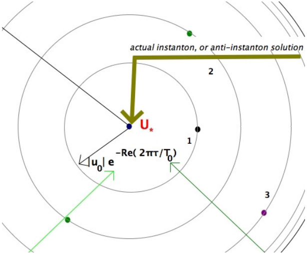
Fine structure of the saddle points



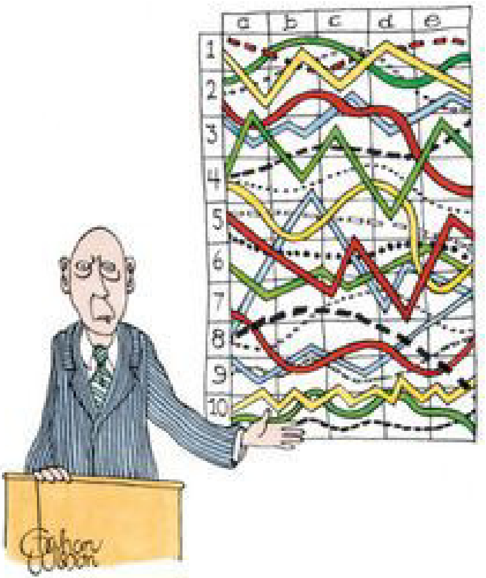
Where are the instantons now?



Where are the instantons/antiinstantons now?

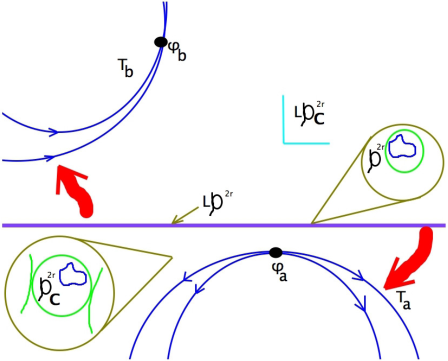


Degenerate abelian variety. The solution requires $T = \infty$.



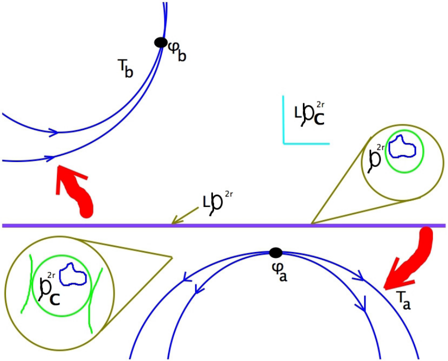
"I'll pause for a moment so you can let this information sink in."

Next steps



- Zero-modes: the whole abelian variety.
  Only middle-dimensional cycle contributes to $T_{\mathbf{a}}$

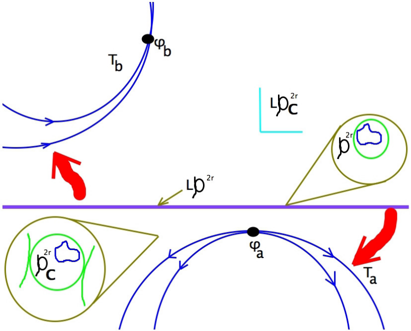
Next steps



- Zero-modes: the whole abelian variety.

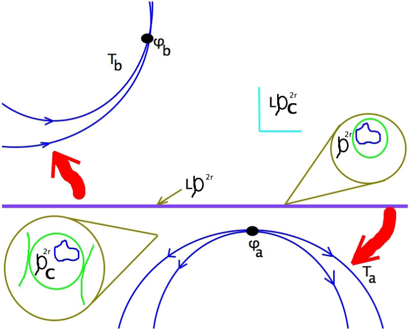
  Only middle-dimensional cycle contributes to $T_{\mathbf{a}}$
- Non-zero modes: Evaluate the one-loop determinants

Next steps



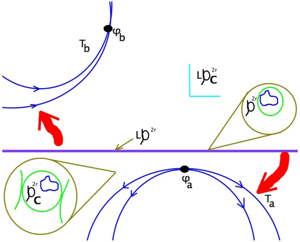
- Zero-modes: the whole abelian variety.
  Only middle-dimensional cycle contributes to $T_{\mathbf{a}}$
- Non-zero modes: Evaluate the one-loop determinants
- Figure out relative phases of $\varphi_{\mathbf{a}}$ contributions (spectral flow)

Next steps



- Zero-modes: the whole abelian variety.
  Only middle-dimensional cycle contributes to $T_{\mathbf{a}}$
- Non-zero modes: Evaluate the one-loop determinants
- Relative phases of $\varphi_{\mathbf{a}}$ contributions:
  the imprint of the "negative" modes
- Set up perturbation theory to include $\hbar$-corrections

Next steps



- Zero-modes: the whole abelian variety.
  Only middle-dimensional cycle contributes to $T_{\mathbf{a}}$
- Non-zero modes: Evaluate the one-loop determinants
- Relative phases of $\varphi_{\mathbf{a}}$ contributions:
  the imprint of the "negative" modes
- Set up perturbation theory to include $\hbar$-corrections
- Recognize in the asymptotic nature of $\hbar$-expansion
  the influence of different $\varphi_{\mathbf{a}}$'s, e.g.
- in the poles of the Borel transforms

Resurgence

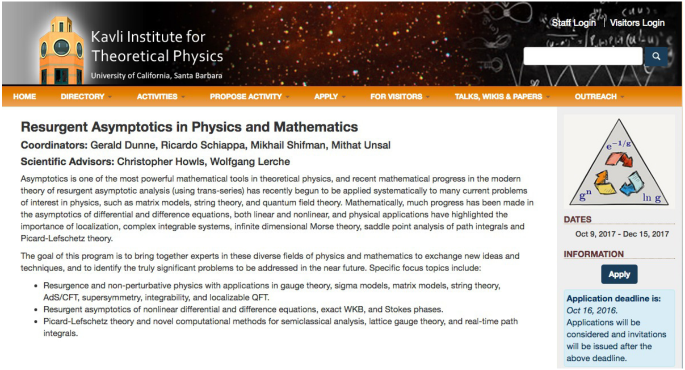
connects perturbative and non-perturbative physics

Resurgence connects perturbative and non-perturbative physics

Resurgence, perturbative/non-perturbative relations

J. Ecalle'81
A. Voros'81-04
F .Pham'83-97
A. Vainshtein'64
C. Bender and T. Wu'69
J.J. Duistermaat and V.W. Guillemin'75
L. Lipatov'77
B. Malgrange'79
M. Shifman, A. Vainshtein, V. Zakharov'83
E Bogomolny, J. Zinn-Justin'84
M.V. Berry and C.J. Howls'94
P. Argyres, M. Unsal'12
M. Kontsevich and Y. Soibelman'??

Resurgence



Kavli Institute for Theoretical Physics
University of California, Santa Barbara

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HOME DIRECTORY ACTIVITIES PROPOSE ACTIVITY APPLY FOR VISITORS TALKS, WIKIS & PAPERS OUTREACH

Resurgent Asymptotics in Physics and Mathematics

Coordinators: Gerald Dunne, Ricardo Schiappa, Mikhail Shifman, Mithat Unsal

Scientific Advisors: Christopher Howls, Wolfgang Lerche

Asymptotics is one of the most powerful mathematical tools in theoretical physics, and recent mathematical progress in the modern theory of resurgent asymptotic analysis (using trans-series) has recently begun to be applied systematically to many current problems of interest in physics, such as matrix models, string theory, and quantum field theory. Mathematically, much progress has been made in the asymptotics of differential and difference equations, both linear and nonlinear, and physical applications have highlighted the importance of localization, complex integrable systems, infinite dimensional Morse theory, saddle point analysis of path integrals and Picard-Lefschetz theory.

The goal of this program is to bring together experts in these diverse fields of physics and mathematics to exchange new ideas and techniques, and to identify the truly significant problems to be addressed in the near future. Specific focus topics include:

- Resurgence and non-perturbative physics with applications in gauge theory, sigma models, matrix models, string theory, AdS/CFT, supersymmetry, integrability, and localizable QFT.
- Resurgent asymptotics of nonlinear differential and difference equations, exact WKB, and Stokes phases.
- Picard-Lefschetz theory and novel computational methods for semiclassical analysis, lattice gauge theory, and real-time path integrals.

DATES

Oct 9, 2017 - Dec 15, 2017

INFORMATION

Apply

Application deadline is:
Oct 16, 2016.
Applications will be considered and invitations will be issued after the above deadline.

Origin of these ideas

Bethe/gauge correspondence

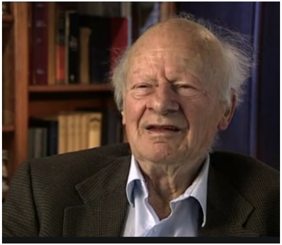
Gauge theories with $\mathcal{N} = (2,2)$ $d = 2$ super-Poincare invariance

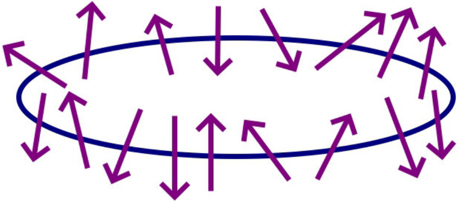
$\Leftrightarrow$

Quantum integrable systems

$\diamondsuit$

QIS ≈ Bethe Ansatz soluble





Bethe/gauge correspondence

NN, S.Shatashvili, circa 2007

Supersymmetric vacua (in finite volume) of gauge theory

$\Leftrightarrow$

Stationary states of the QIS

Bethe/gauge correspondence

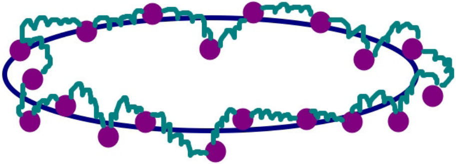
Equations for vacua from minimization of the effective potential

$$\frac{\partial \widetilde{\mathcal{W}}(\sigma)}{\partial \sigma_i} = 2\pi \mathrm{i} n_i \,, \qquad i = 1, \ldots, r$$

$\Leftrightarrow$

Bethe equations of the QIS

Quantum mechanics from 4d gauge theory



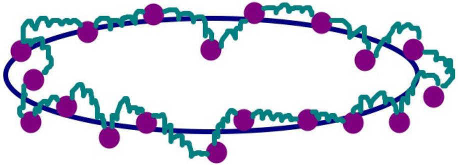
Four dimensional theories

e.g. $\mathcal{N} = 2$ super-Yang-Mills theory in four dimensions

Viewed as two dimensional theories with $SO(2)$ R-symmetry

rotations of two extra dimensions

Quantum mechanics from 4d gauge theory



Four dimensional $\mathcal{N} = 2$ theory

Viewed as two dimensional theory with $SO(2)$ R-symmetry
Turn on the twisted mass for this symmetry $\Longrightarrow \hbar$

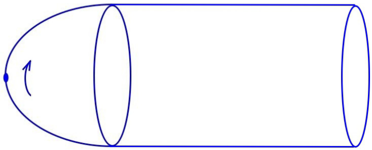
NN, S.Shatashvili, 2009

Compactify the $1+1$ dimensional spacetime on $\mathbb{R} \times \mathbb{S}^1$ (finite volume)

Quantum mechanics from 4d gauge theory

Four dimensional $\mathcal{N} = 2$ theory

Compactified onto $\mathcal{D}_\hbar \times \mathbb{S}^1 \times \mathbb{R}^1$ (cigar $\times$ circle $\times$ time axis)



θ-angular coordinate on $\mathcal{D}_\hbar$

With Ω-deformation along the cigar $\mathcal{D} = D_\mu \phi \longrightarrow D_\mu \phi + \hbar F_{\mu\theta}$

Quantum mechanics from 4d gauge theory

Four dimensional $\mathcal{N} = 2$ theory

Compactified onto $\mathcal{D}_\hbar \times \mathbb{S}^1 \times \mathbb{R}^1$ (cigar $\times$ circle $\times$ time axis)

With Ω-deformation along the cigar $\mathcal{D}$

At low energy

Quantum mechanics from 4d gauge theory

Four dimensional $\mathcal{N} = 2$ theory

Compactified onto $\mathcal{D}_\hbar \times \mathbb{S}^1 \times \mathbb{R}^1$ (cigar $\times$ circle $\times$ time axis)

$\times \mathbb{S}^1 \times \mathbb{R}^1$

at low energy $\downarrow$

$\times \mathbb{R}^1$

Becomes 2d sigma model on $\mathbb{R}_+ \times \mathbb{R}^1$

Quantum mechanics from 4d gauge theory

Four dimensional $\mathcal{N} = 2$ theory

Compactified onto $\mathcal{D}_\hbar \times \mathbb{S}^1 \times \mathbb{R}^1$ (cigar $\times$ circle $\times$ time axis)

$\times \mathbb{S}^1 \times \mathbb{R}^1$

at low energy $\downarrow$

$\times \mathbb{R}^1$

Becomes 2d sigma model on $\mathbb{R}_+ \times \mathbb{R}^1 \implies$ deformation quantization

introduced in 1978
by F. Bayen, L. Boutet de Monvel, M. Flato,
C. Fronsdal, A. Lichnerowicz et D. Sternheimer'78,
existence of formal def.quant. shown by M. Kontsevich in 1999
sigma model explored by A. Cattaneo and G. Felder'99

NN, E.Witten'2009
Using A.Kapustin,D.Orlov's branes'2003

Quantum mechanics from 4d gauge theory

Partition function of the quantum system

$$\mathrm{Tr}_{\mathcal{H}_{\rm qis}}\, e^{-\frac{1}{\hbar}\sum_k \tau_k \widehat{H}_k}$$

Quantum mechanics from 4d gauge theory

Partition function of the quantum system

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{qis}}}\, e^{-\frac{1}{\hbar}\sum_k \tau_k \widehat{H}_k} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{vac}}}\, e^{-\frac{1}{\hbar}\sum_k \tau_k \mathcal{O}_k}$$

with τ_k the set of "times" - generalized Gibbs ensemble

with $\mathcal{O}_k$ the basis of the twisted chiral ring

Quantum mechanics from 4d gauge theory

Partition function of the quantum system

$$\mathrm{Tr}_{\mathcal{H}_{\rm qis}}\, e^{-\frac{1}{\hbar}\sum_k \tau_k \widehat{H}_k} = \mathrm{Tr}_{\mathcal{H}_{\rm vac}}\, e^{-\frac{1}{\hbar}\sum_k \tau_k \mathcal{O}_k} = \mathrm{Tr}_{\mathcal{H}_{\rm vac}}\, (-1)^F\, e^{-\frac{1}{\hbar}\sum_k \tau_k \mathcal{O}_k}$$

assuming all vacua are bosonic

Quantum mechanics from 4d gauge theory

Partition function of the quantum system

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{qis}}}\, e^{-\frac{1}{\hbar}\sum_k \tau_k \widehat{H}_k} = \mathrm{Tr}_{\mathcal{H}_{\mathrm{vac}}}\,(-1)^F\, e^{-\frac{1}{\hbar}\sum_k \tau_k \mathcal{O}_k} =$$

$$= \mathrm{Tr}_{\mathcal{H}_{\mathrm{gauge}}}\,(-1)^F\, e^{-\frac{1}{\hbar}\sum_k \tau_k \mathcal{O}_k}$$

using $[\mathcal{Q}, \mathcal{O}_k] = 0$ and the usual Witten index argument

Quantum mechanics from 4d gauge theory

Partition function of the quantum system

$$=$$

Partition function of the $\mathcal{N} = 2$ gauge theory on $\mathbb{T}^2 \times \mathcal{D}$
with Ω-deformation along $\mathcal{D}$

$$D_\mu \phi \longrightarrow D_\mu \phi + \hbar F_{\mu\theta}$$

and 2-observables of $\mathcal{O}_k$ integrated along $\mathcal{D}$

$$\frac{1}{\hbar}\mathcal{O}_k = \int_{\mathcal{D}} \mathcal{O}_k^{(2)}$$

The latter description makes sense even when $\hbar \to 0$

Partition function of the quantum-mechanical system

$$=$$

susy Partition function of the $\mathcal{N} = 2$ gauge theory

on $\mathbb{T}^2 \times \mathcal{D}$

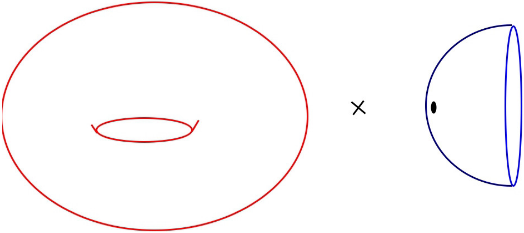
$$\int_{4d\ \ gauge\ \ superfields} e^{-\int_{\mathbb{T}^2 \times \mathcal{D}} \mathcal{L}_{\mathrm{SYM}}} \ e^{\sum_k \tau_k \int_{\mathcal{D}} \mathcal{O}_k^{(2)}}$$

$\sim$Donaldson's surface-observables $\uparrow\uparrow\uparrow$ along $\mathcal{D}$

Unification: effective superpotential

Claim: the $\mathcal{N} = 2$ Landau-Ginzburg description follows from $\mathcal{N} = 2$ gauge theory!

Compactify the theory on large $\mathbb{T}^2$



$\mathcal{N} = 2$ Landau-Ginzburg description

follows from low-energy effective $\mathcal{N} = 2$ gauge theory!

Compactify the theory on large $\mathbb{T}^2$ (compared to Λ_{QCD} scale)

take into account the electric **n** and magnetic **m** fluxes

go to extreme infrared

$$S_{\text{eff}} = \int_{\mathcal{D}} \mathcal{W}^{(2)}_{\mathbf{n},\mathbf{m}} + \ \mathrm{D} - \mathrm{terms}$$

$\mathcal{N} = 2$ Landau-Ginzburg description

follows from low-energy effective $\mathcal{N} = 2$ gauge theory!

Compactify the theory on large $\mathbb{T}^2$

take into account the electric **n** and magnetic **m** fluxes

$$\mathcal{W}_{\mathbf{n},\mathbf{m}} = \sum_{j=1}^{r} n_j a^j + m^j a_{D,j} - \mathrm{i}\tau_j u_j$$

Losev, NN, Shatashvili'97, '98, '99, rigid $\mathcal{N} = 2$, $d = 2$
Vafa, Taylor'99 $\tau = 0$, noncompact CY3, $\mathcal{N} = 1$, $d = 4$
Gukov,Vafa, Witten'99 $\tau = 0$, CY4, $\mathcal{N} = 2$, $d = 2$ sugra

From quantum mechanics to quantum field theory

What we have learned

From quantum mechanics to quantum field theory

From what we have learned it is clear, we should be looking for

Complex solutions of equations of motion

on spacetime of the form

$$S^1_T \times M_d$$

Complexify the phase space of the theory on M_d

If we are lucky it will be an ∞-dimensional

algebraic integrable system

Complexify the phase space of the theory on M_d

Even if we are unlucky we may still find

the complex energy levels to have non-trivial π_1

$\Longrightarrow$ non-trivial critical points

Specific examples

$\mathbb{CP}^1$-model

$$S = R^2 \int_\Sigma d^2\sigma\, \partial_a \vec{n} \cdot \partial_a \vec{n}\,, \qquad \vec{n}\cdot\vec{n} = 1, \quad \vec{n} \in \mathbb{R}^3$$

Specific examples

$\mathbb{CP}^1$-model

Now make $\vec{n} \in \mathbb{C}^3$

Equations of motion read

$$\left(-\partial\bar{\partial} + u\right)\vec{n} = 0$$

$$u = \partial\vec{n} \cdot \bar{\partial}\vec{n}$$

$\mathbb{CP}^1$-model with $\vec{n} \in \mathbb{C}^3$

Equations of motion:

$$\left(-\partial\bar{\partial} + u\right) \vec{n} = 0$$

$T = \partial\vec{n} \cdot \partial\vec{n}$ - holo $(2,0)$-diff on Σ, $\bar{\partial} T = 0$
$\tilde{T} = \bar{\partial}\vec{n} \cdot \bar{\partial}\vec{n}$ - antiholo $(0,2)$-diff on Σ, $\partial \tilde{T} = 0$
$u = \partial\vec{n} \cdot \bar{\partial}\vec{n}$: *consistent* Schrodinger potential

I. Krichever, $\Sigma = T^2$, $T = \tilde{T} = 0$, '94

$\mathbb{CP}^1$-model with $\vec{n} \in \mathbb{C}^3$

Equations of motion:

$$\left(-\partial\bar{\partial} + u\right) \vec{n} = 0$$

$T = \partial\vec{n} \cdot \partial\vec{n}$ - holo $(2,0)$-diff on Σ, $\bar{\partial} T = 0$
$\tilde{T} = \bar{\partial}\vec{n} \cdot \bar{\partial}\vec{n}$ - antiholo $(0,2)$-diff on Σ, $\partial \tilde{T} = 0$

$\uparrow$ Conservation laws

$\mathbb{CP}^1$-model with $\vec{n} \in \mathbb{C}^3$

Equations of motion:

$$\left(-\partial\bar{\partial} + u\right) \vec{n} = 0$$

When $\Sigma = \mathbb{T}^2,\ T = t dz^2,\ \tilde{T} = \tilde{t} d\bar{z}^2$

$$z \sim z + m + n\tau$$

With some constants $t, \tilde{t} \in \mathbb{C}$

$\mathbb{CP}^1$-model with $\vec{n} \in \mathbb{C}^3$

To exhibit the algebraic integrability
one defines an analytic curve $\mathcal{C}$
so that its Jacobian (or Prym variety) is
abelian variety on which the motion linearizes

Fermi-surface curve

$$\mathcal{C}_{Fermi} \subset \mathbb{C}^{\times} \times \mathbb{C}^{\times}$$

$$\left(-\partial\bar{\partial} + u(z,\bar{z})\right)\psi = 0,$$

Periodic potential: $u(z+1,\bar{z}+1) = u(z+\tau,\bar{z}+\bar{\tau}) = u(z,\bar{z})$

Bloch boundary conditions

$$\psi(z+1,\bar{z}+1) = a\,\psi(z,\bar{z}),$$

$$\psi(z+\tau,\bar{z}+\bar{\tau}) = b\,\psi(z,\bar{z})$$

Time evolution is hidden

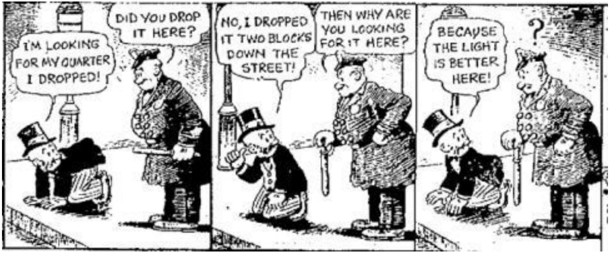
In progress, with I. Krichever

$SU(2)$-gauge theory in $3 + 1$

$SU(2)$-gauge theory in $3+1$

put the theory on $S^1_T \times S^3_{\mathrm{space}}$

$SU(2)$-gauge theory on ...



Impose rotational invariance!

Start with $SU(2)$-gauge theory on $\mathbb{R}^1_T \times \mathbb{R}^3_{\text{space}}$

$$ds^2 = dt^2 + dr^2 + r^2 d\Omega_2^2$$

Classical Yang-Mills is conformally invariant $\Longrightarrow AdS_2 \times S^2$

$$d\tilde{s}^2 = \frac{dt^2 + dr^2}{r^2} + d\Omega_2^2$$

Cylindrical symmetric ansatz (space $SO(3)$ locked with color $SU(2)$)

Start with $SU(2)$-gauge theory on $\mathbb{R}^1_T \times \mathbb{R}^3_{\text{space}}$

$$ds^2 = dt^2 + dr^2 + r^2 d\Omega_2^2$$

Classical Yang-Mills is conformally invariant $\Longrightarrow AdS_2 \times S^2$

$$d\tilde{s}^2 = \frac{dt^2 + dr^2}{r^2} + d\Omega_2^2$$

$\nearrow$

Cylindrical symmetric ansatz (space $SO(3)$ locked with internal $SU(2)$)

$SU(2)$-gauge theory on $\mathbb{R}^1_T \times \mathbb{R}^3_{\text{space}}$

Cylindrical symmetric ansatz, $\hat{n} \in S^2$

cf. L. Faddeev, A. Niemi'99, $\hat{n}$ dynamical

$$A = \hat{\sigma} \cdot \hat{n}\, a + (1+\phi_2)\hat{n} \cdot (\hat{\sigma} \times d\hat{n}) + \phi_1 \hat{\sigma} \cdot d\hat{n}$$

S^2-dependence drops

We are left with

the $U(1)$ gauge field a

a complex scalar $\phi = \phi_1 + \mathrm{i}\phi_2$

On AdS_2 spacetime

$$S_{YM} \to \int_{AdS_2} da \wedge \star da + D_a\phi \wedge \star D_a\bar{\phi} + \sqrt{g}\,\left(1 - |\phi|^2\right)^2$$

Witten'78

In our case: $SU(2)$-gauge theory on $S^1_T \times S^3_{\text{space}}$

$$ds^2 = dt^2 + R^2\left(d\theta^2 + \cos(\theta)^2 d\Omega_2^2\right)$$

Classical Yang-Mills is conformally invariant $\Longrightarrow AdS_2 \times S^2$

$$d\tilde{s}^2 = \frac{d(t/R)^2 + d\theta^2}{\cos(\theta)^2} + d\Omega_2^2$$

In our case: $SU(2)$-gauge theory on $S^1_T \times S^3_{\rm space}$

We can again use the cylindrical symmetric ansatz

Again the S^2-dependence drops

Again we are left with

the $U(1)$ gauge field a and a complex scalar $\phi = \phi_1 + \mathrm{i}\phi_2$

On AdS_2

Global identifications are now different...

Similarity to the anharmonic oscillator looks promising...

...to be continued

String theory?

Complex saddle points: non-unitary 2d CFT's
RG flows in the space of complexified couplings
Lefschetz thimbles?
Proper framework for theories with complex c_L, c_R central charges?
4d $\mathcal{N} = 2$ gauge theories! again

Remark on space-time dimensionality and susy

We saw that non-supersymmetric quantum mechanics, i.e. $0+1$ theory when subject to the full analytic continuation in all couplings

Remark on space-time dimensionality and susy

We saw that non-supersymmetric quantum mechanics, i.e. $0+1$ theory
when subject to the full analytic continuation in all couplings
Embeds naturally into a supersymmetric gauge theory in $3+1$ dimensions

Remark on space-time dimensionality and susy

What is the case of

a non-supersymmetric theory in $3+1$ dimensions

subject to the full analytic continuation in all couplings?

Remark on space-time dimensionality and supersymmetry

What is the case of

a non-supersymmetric theory in $3+1$ dimensions

subject to the full analytic continuation in all couplings?

Some gauge(?) theory in $6+1$?

Remark on space-time dimensionality and susy

What is the case of

a non-supersymmetric theory in $3+1$ dimensions

subject to the full analytic continuation in all couplings?

Chern-Simons theory of the $(2,0)$ superconformal theory in six dimensions?

Remark on space-time dimensionality and susy

What is the case of

a non-supersymmetric theory in $3+1$ dimensions

subject to the full analytic continuation in all couplings?

Chern-Simons theory of the $(2,0)$ superconformal theory in six dimensions?

Could the supersymmetry in the bulk
nearly cancel the cosmological constant
without affecting the Einstein gravity in our
effectively $3+1$ dimensions?

THANK YOU