Dirac fermions in arbitrary external classical fields: quantum spin dynamics

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Outline

1. Relativistic particles in curved spacetimes
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   - Geometry of spacetime
2. Spin $\frac{1}{2}$ particle in curved spacetime
   - Dirac Hamiltonian for arbitrary metric
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   - Foldy-Wouthuysen Hamiltonian and equations of motion
3. Classical and quantum spin: comparison
   - Classical spin in external fields
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Dynamics of spin and equivalence principle

- High-energy experiments take place in curved space or in noninertial frame (for example, on Earth)
- Equivalence principle (EP) - a cornerstone of gravity
- Newton's theory $\Rightarrow$ Einstein's "falling elevator"
- Colella-Overhauser-Werner (1975) and Bonse-Wroblewski experiments - EP for quantum-mechanical systems:
- Measured phase shift due to inertial and gravitational force
- Gravity on *spin*: EP for relativistic particles?
- Classical theory of spin: Frenkel (1928), Mathisson (1937), Papapetrou (1951), Weyssenhoff-Raabe (1947)
- Compare classical rotator and quantum spin
- Relativistic spin effects not measured yet!

Arbitrary Riemannian geometry in 4 dimensions

- Let t be time, x^a $(a = 1, 2, 3)$ be spatial coordinates:

$$ds^2 = V^2 c^2 dt^2 - \delta_{\widehat{a}\widehat{b}} W^{\widehat{a}}{}_c W^{\widehat{b}}{}_d \,(dx^c - K^c cdt)\,(dx^d - K^d cdt)$$

  V and K^a, and 3×3 matrix $W^{\widehat{a}}{}_b$ depend arbitrarily on t, x^a.
- Their number $1 + 3 + 9 = 13$ but rotation $W^{\widehat{a}}{}_b \longrightarrow L^{\widehat{a}}{}_{\widehat{c}} W^{\widehat{c}}{}_b$ is allowed with arbitrary $L^{\widehat{a}}{}_{\widehat{c}}(t, x) \in SO(3)$: $\Longrightarrow 13 - 3 = 10$
- Coframe e^α_i with $g_{\alpha\beta} e^\alpha_i e^\beta_j = g_{ij}$, $g_{\alpha\beta} = \mathrm{diag}(c^2, -1, -1, -1)$:

$$e^{\widehat{0}}_i = V\,\delta^0_i, \qquad e^{\widehat{a}}_i = W^{\widehat{a}}{}_b \left(\delta^b_i - cK^b\,\delta^0_i\right), \qquad a = 1, 2, 3$$

- Exact metric of flat spacetime in noninertial frame

$$V = 1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}, \quad W^{\widehat{a}}{}_b = \delta^a_b, \quad K^a = -\frac{1}{c}\,(\boldsymbol{\omega}\times\boldsymbol{r})^a$$

Dirac particle in gravitational & electromagnetic field

- Fermion with moments (AMM $\mu' = \frac{(g-2)e\hbar}{4m}$ & EDM $\delta' = \frac{be\hbar}{2mc}$)

$$\left(i\hbar\gamma^\alpha D_\alpha - mc + \frac{\mu'}{2c}\sigma^{\alpha\beta}F_{\alpha\beta} + \frac{\delta'}{2}\sigma^{\alpha\beta}G_{\alpha\beta}\right)\psi = 0$$

- Spinor covariant derivative (with $\sigma_{\alpha\beta} = i\gamma_{[\alpha}\gamma_{\beta]}$)

$$D_\alpha = e^i_\alpha D_i, \qquad D_i = \partial_i - \frac{ie}{\hbar}A_i + \frac{i}{4}\sigma^{\alpha\beta}\Gamma_{i\,\alpha\beta}$$

- Connection for general spacetime geometry

$$\begin{aligned}\Gamma_{i\,\widehat{a}\widehat{0}} &= \frac{c^2}{V}W^b{}_{\widehat{a}}\,\partial_b V\,e_i{}^{\widehat{0}} - \frac{c}{V}\,\mathcal{Q}_{(\widehat{a}\widehat{b})}\,e_i{}^{\widehat{b}},\\ \Gamma_{i\,\widehat{a}\widehat{b}} &= \frac{c}{V}\,\mathcal{Q}_{[\widehat{a}\widehat{b}]}\,e_i{}^{\widehat{0}} + \left(\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} + \mathcal{C}_{\widehat{a}\widehat{c}\widehat{b}} + \mathcal{C}_{\widehat{c}\widehat{b}\widehat{a}}\right)e_i{}^{\widehat{c}}\end{aligned}$$

- Here anholonomity $\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = W^d{}_{\widehat{a}}W^e{}_{\widehat{b}}\,\partial_{[d}W^{\widehat{c}}{}_{e]}$ and

$$\mathcal{Q}_{\widehat{a}\widehat{b}} = g_{\widehat{a}\widehat{c}}W^d{}_{\widehat{b}}\left(\frac{1}{c}\dot{W}^{\widehat{c}}{}_d + K^e\partial_e W^{\widehat{c}}{}_d + W^{\widehat{c}}{}_e\partial_d K^e\right)$$

Dirac Hamiltonian

- Naive Hamiltonian is not Hermitian. Rescale wave function $\psi \longrightarrow \left(\sqrt{-g}e_{\widehat{0}}^{0}\right)^{\frac{1}{2}} \psi$ and recast Dirac wave equation into Schrodinger form $i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi$

Dirac Hamiltonian (with $\mathcal{F}^b{}_a = VW^b{}_{\widehat{a}}$ and $\boldsymbol{\pi} = -i\hbar\boldsymbol{\nabla} - e\boldsymbol{A}$)

$$\mathcal{H} = \beta mc^2 V + e\Phi + \frac{c}{2}\left(\pi_b\,\mathcal{F}^b{}_a\alpha^a + \alpha^a\mathcal{F}^b{}_a\pi_b\right)$$
$$+\frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{K}\right) + \frac{\hbar c}{4}\left(\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma} - \Upsilon\gamma_5\right)$$
$$-\beta V\left(\boldsymbol{\Sigma}\cdot\boldsymbol{\mathcal{M}} + i\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{P}}\right)$$

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- Here $\beta = \gamma^{\widehat{0}}$, $\alpha^a = \gamma^{\widehat{0}}\gamma^{\widehat{a}}$, $\gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$,

$$\Upsilon = V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{a}\widehat{b}\widehat{c}} = -V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}}, \qquad \Xi_{\widehat{a}} = \frac{V}{c}\,\epsilon_{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{0}}{}^{\widehat{b}\widehat{c}} = \epsilon_{\widehat{a}\widehat{b}\widehat{c}}\,\mathcal{Q}^{\widehat{b}\widehat{c}}$$

Electrodynamics in curved spacetime

- Gravity is universal: affects also electromagnetism. How?
- Basic objects: field strength F, excitation H and current J

Maxwell's theory – without coordinates and frames

$$dF = 0, \qquad dH = J, \qquad H = \lambda_0 \star F, \qquad \lambda_0 = \sqrt{\varepsilon_0/\mu_0}$$

- Coordinates x^i: $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$, $H = \frac{1}{2}H_{ij}dx^i \wedge dx^j$, and $J = \frac{1}{6}J_{ijk}dx^i \wedge dx^j \wedge dx^k$ are $(1+3)$ decomposed:

$$\boldsymbol{E}_a = \{F_{10}, F_{20}, F_{30}\}, \qquad \boldsymbol{B}^a = \{F_{23}, F_{31}, F_{12}\}$$

$$\boldsymbol{H}_a = \{H_{01}, H_{02}, H_{03}\}, \qquad \boldsymbol{D}^a = \{H_{23}, H_{31}, H_{12}\}$$

- $$\boldsymbol{J}^a = \{-J_{023}, -J_{031}, -J_{012}\}, \qquad \rho = J_{123}$$

- Maxwell equations are recast into standard form

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0,$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} - \dot{\boldsymbol{D}} = \boldsymbol{J}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$$

- Gravity/inertia encoded in *constitutive relation* $H = H(F)$

$$D^a = \frac{\varepsilon_0 w}{V}\,\underline{g}^{ab}E_b - \lambda_0\frac{w}{V}\,\underline{g}^{ad}\epsilon_{bcd}K^c\,B^b,$$

$$H_a = \frac{1}{\mu_0 w V}\left\{(V^2 - K^2)\underline{g}_{ab} + K_aK_b\right\}B^b - \lambda_0\frac{w}{V}\,\epsilon_{adc}K^c\,\underline{g}^{db}E_b$$

Here $K_a = \underline{g}_{ab}K^b$, $K^2 = \underline{g}_{ab}K^aK^b$ and $w = \det W^{\widehat{c}}{}_d$.

- Frame e_i^α needed for fermions $\Longrightarrow F_{\alpha\beta} = e^i_\alpha e^j_\beta F_{ij}$
- Components: $\mathfrak{E}_a = \{F_{\widehat{1}\widehat{0}}, F_{\widehat{2}\widehat{0}}, F_{\widehat{3}\widehat{0}}\}$ & $\mathfrak{B}^a = \{F_{\widehat{2}\widehat{3}}, F_{\widehat{3}\widehat{1}}, F_{\widehat{1}\widehat{2}}\}$
- Relation between holonomic and anholonomic fields

$$\mathfrak{E}_a = \frac{1}{V}\,W^b{}_{\widehat{a}}(\boldsymbol{E} + c\boldsymbol{K}\times\boldsymbol{B})_b, \quad \mathfrak{B}^a = \frac{1}{w}\,W^{\widehat{a}}{}_b\,\boldsymbol{B}^b$$

- Nonminimal coupling $-\beta V\,(\boldsymbol{\Sigma}\cdot\boldsymbol{\mathcal{M}} + i\boldsymbol{\alpha}\cdot\boldsymbol{\mathcal{P}})$ governed by

$$\boldsymbol{\mathcal{M}}^a = \mu'\mathfrak{B}^a + \delta'\mathfrak{E}^a, \qquad \boldsymbol{\mathcal{P}}_a = c\delta'\mathfrak{B}_a - \mu'\mathfrak{E}_a/c.$$

- Foldy-Wouthuysen transform needed to reveal physics
- FW Hamiltonian $\mathcal{H}_{FW} = \mathcal{H}_{FW}^{(1)} + \mathcal{H}_{FW}^{(2)}$. E.g. *for Earth:*
- with $\epsilon = \sqrt{m^2c^4 + \boldsymbol{\pi}^2c^2}$ and $\boldsymbol{\mathcal{J}} = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}} + \frac{\partial\boldsymbol{\mathcal{P}}}{c\partial t}$ we find

$$\mathcal{H}_{FW}^{(1)} = \beta\epsilon + q\Phi - \boldsymbol{\omega}\cdot(\boldsymbol{r}\times\boldsymbol{\pi}) - \frac{\hbar}{2}\boldsymbol{\omega}\cdot\boldsymbol{\Sigma} - \frac{q\hbar c^2}{4}\left\{\frac{1}{\epsilon}, \boldsymbol{\Pi}\cdot\boldsymbol{\mathfrak{B}}\right\} + \frac{q\hbar c^2}{8}\left\{\frac{1}{\epsilon(\epsilon+mc^2)}, \left[\boldsymbol{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{\mathfrak{E}} - \boldsymbol{\mathfrak{E}}\times\boldsymbol{\pi}) - \hbar\boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{E}}\right]\right\},$$

$$\mathcal{H}_{FW}^{(2)} = -\frac{c}{4}\left\{\frac{1}{\epsilon}, \left[\boldsymbol{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{\mathcal{P}} - \boldsymbol{\mathcal{P}}\times\boldsymbol{\pi}) - \hbar\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}}\right]\right\} - \boldsymbol{\Pi}\cdot\boldsymbol{\mathcal{M}} + \frac{c^2}{4}\left\{\frac{1}{\epsilon(\epsilon+mc^2)}, \left[(\boldsymbol{\Pi}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{\mathcal{M}}) + (\boldsymbol{\mathcal{M}}\cdot\boldsymbol{\pi})(\boldsymbol{\Pi}\cdot\boldsymbol{\pi}) + \beta\frac{\hbar}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\mathcal{J}} + \boldsymbol{\mathcal{J}}\cdot\boldsymbol{\pi}) - \beta\frac{\hbar}{2c}\left\{\left([\boldsymbol{\omega}\times\boldsymbol{r}]\cdot\boldsymbol{\nabla}\right), (\boldsymbol{\pi}\cdot\boldsymbol{\mathcal{P}})\right\}\right]\right\}$$

- Here $\{\ ,\ \}$ anticommutators, $\mathcal{T} = 2\epsilon^2 + \{\epsilon, mc^2V\}$, $\boldsymbol{\Pi} = \beta\boldsymbol{\Sigma}$,
- *This result is exact* – no (weak field etc) approximations for $V, W^{\widehat{a}}{}_b, K^a$. Planck $\hbar$ is the only small parameter

Classical spin in external fields

- Dynamics of spinning particle in external classical fields

$$\frac{dU^\alpha}{d\tau} = \mathcal{F}^\alpha, \qquad \frac{dS^\alpha}{d\tau} = \Phi^\alpha{}_\beta S^\beta$$

- Physical spin is defined in rest frame of particle $u^\alpha = \delta^\alpha_0$
- Local Lorentz transformation $U^\alpha = \Lambda^\alpha{}_\beta u^\beta$

$$\Lambda^\alpha{}_\beta = \left(\begin{array}{c|c} \gamma & \gamma v_b/c^2 \\ \hline \gamma v^a & \delta^a_b + (\gamma - 1)v^a v_b/v^2 \end{array}\right)$$

Dynamics of physical spin $s^\alpha = (\Lambda^{-1})^\alpha{}_\beta S^\beta$

$$\frac{ds^\alpha}{d\tau} = \Omega^\alpha{}_\beta s^\beta,$$

$$\Omega^\alpha{}_\beta = (\Lambda^{-1})^\alpha{}_\gamma \Phi^\gamma{}_\delta \Lambda^\delta{}_\beta - (\Lambda^{-1})^\alpha{}_\gamma \frac{d}{d\tau}\Lambda^\gamma{}_\beta$$

Mathisson-Papapetrou theory

- In curved spacetime and electromagnetic field

$$\mathcal{F}^\alpha = -\Gamma_{\gamma\beta}{}^{\alpha}\, u^\gamma u^\beta - \frac{e}{m}\, g^{\alpha\beta} F_{\beta\gamma} u^\gamma,$$

$$\Phi^\alpha{}_\beta = -\Gamma_{\gamma\beta}{}^{\alpha}\, u^\gamma - \frac{e}{m}\, g^{\alpha\gamma} F_{\gamma\beta} - \frac{2}{\hbar}\left[M^\alpha{}_\beta + \frac{1}{c^2}\left(M_{\beta\gamma}u^\alpha u^\gamma - M^{\alpha\gamma}u_\beta u_\gamma\right)\right]$$

- Spin is affected by force due to “polarization” tensor

$$M_{\alpha\beta} = \mu'\, F_{\alpha\beta} + c\delta'\, F^*_{\alpha\beta}.$$

- Dimensionless parameters $a = \frac{g-2}{2}$ and b characterize magnitude of AMM and EDM: $\mu' = a\,\frac{e\hbar}{2m}$ and $\delta' = b\,\frac{e\hbar}{2mc}$
- In $(1+3)$-decomposed form, we recover

$$M_{\widehat{0}\widehat{a}} = c\mathcal{P}_a, \qquad M_{\widehat{a}\widehat{b}} = \epsilon_{abc}\mathcal{M}^c$$

- Physical spin $\boldsymbol{s}$ precesses wrt rest frame: $\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{s}$

Spin dynamics on Earth (with $\boldsymbol{g} = -\frac{GM}{r^3}\boldsymbol{r}$, $\gamma = 1/\sqrt{1 - v^2/c^2}$)

$$\begin{aligned}
\boldsymbol{\Omega} \;=\; & \frac{e}{m}\left\{-\frac{1}{\gamma}\mathfrak{B} + \frac{1}{\gamma+1}\frac{\boldsymbol{v}\times\mathfrak{E}}{c^2}\right\} - \boldsymbol{\omega} + \frac{2\gamma+1}{\gamma+1}\frac{\boldsymbol{v}\times\boldsymbol{g}}{c^2} \\
& - \frac{2\mu'}{\hbar}\left\{\mathfrak{B} - \frac{\boldsymbol{v}\times\mathfrak{E}}{c^2} - \frac{\gamma}{\gamma+1}\boldsymbol{v}\frac{\mathfrak{B}\cdot\boldsymbol{v}}{c^2}\right\} \\
& - \frac{2\delta'}{\hbar}\left\{\mathfrak{E} + \boldsymbol{v}\times\mathfrak{B} - \frac{\gamma}{\gamma+1}\boldsymbol{v}\frac{\mathfrak{E}\cdot\boldsymbol{v}}{c^2}\right\}
\end{aligned}$$

- Analysis of manifestations of terrestrial rotation and gravity in precision high-energy physics: *influence not negligible*
- E.g.: Earth's gravity produces same effect as deuteron's EDM of $\delta' = 1.5 \times 10^{-29}\ e\cdot$cm in planned dEDM experiment with magnetic focusing (AGS proposal EDM Collaboration)

Comparison: quantum vs. classical dynamics

- Quantum (semiclassical) precession velocity

$$\Omega^a_{(1)} = \frac{c^2}{\epsilon}\mathcal{F}^d{}_c p_d\left(\frac{1}{2}\Upsilon\delta^{ac} - \epsilon^{akl}V\mathcal{C}_{kl}{}^c + \frac{\epsilon}{\epsilon + mc^2V}\epsilon^{abc}W^k{}_{\widehat{b}}\,\partial_d V\right),$$

$$\Omega^a_{(2)} = \frac{c}{2}\,\Xi^a - \frac{c^3}{\epsilon(\epsilon + mc^2V)}\,\epsilon^{abc}Q_{(bd)}\delta^{dn}\mathcal{F}^k{}_n p_k \mathcal{F}^l{}_c p_l$$

- Classical precession velocity

$$\Omega^{\widehat{a}} = \frac{\gamma}{V}\left(\frac{1}{2}\Upsilon\, v^{\widehat{a}} - \epsilon^{abc}V\mathcal{C}_{\widehat{b}\widehat{c}}{}^d v_{\widehat{d}} + \frac{\gamma}{\gamma+1}\epsilon^{abc}W^d{}_{\widehat{b}}\,\partial_d V\, v_{\widehat{c}} + \frac{c}{2}\,\Xi^{\widehat{a}} - \frac{\gamma}{\gamma+1}\epsilon^{abc}\mathcal{Q}_{(\widehat{b}\widehat{d})}\frac{v^{\widehat{d}}v_{\widehat{c}}}{c}\right)$$

- 1st comes from Dirac fermion theory; 2nd from Hamilton particle mechanics of Mathisson and Papapetrou

- Quantum (semiclassical) FW Hamiltonian

$$\mathcal{H}_{FW}=\beta\sqrt{m^2c^4V^2+c^2\delta^{cd}\mathcal{F}^a{}_c\mathcal{F}^b{}_d p_a p_b}+c\boldsymbol{K}\cdot\boldsymbol{p}+\frac{\hbar}{2}\boldsymbol{\Pi}\cdot\boldsymbol{\Omega}_{(1)}+\frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{\Omega}_{(2)}$$

- Classical particle with spin

$$\mathcal{H}_{class}=\sqrt{m^2c^4V^2+c^2\delta^{cd}\mathcal{F}^a{}_c\mathcal{F}^b{}_d p_a p_b}+c\boldsymbol{K}\cdot\boldsymbol{p}+s_a\Omega^a$$

- Semiclassical velocity operator from $\frac{i}{\hbar}[\mathcal{H}_{FW},\boldsymbol{x}]$:

$$\beta\,\frac{c^2}{\epsilon}\,\mathcal{F}^b{}_a p_b = v_a \qquad \Longrightarrow \qquad \delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d p_a p_b = \epsilon^2 v^2/c^2$$

- Hence $\epsilon^2 = m^2c^4V^2 + \epsilon^2v^2/c^2$, or $\epsilon = \gamma\, mc^2\, V$.
- This yields a direct correspondence

$$\frac{\epsilon}{\epsilon+mc^2V}=\frac{\gamma}{1+\gamma},\qquad \frac{c^3}{\epsilon(\epsilon+mc^2V)}\,\mathcal{F}^k{}_n p_k\mathcal{F}^l{}_c p_l=\frac{\gamma}{1+\gamma}\,\frac{v_n v_c}{c}$$

- *Perfect agreement of quantum and classical dynamics!*

Conclusions and Outlook

- Relativistic Dirac theory governs quantum dynamics of fermions (also with dipole moments) in curved spacetime
- Earlier results for weak field and stationary configurations [Obukhov, Silenko, Teryaev, Phys. Rev. **D80** (2009) 064044; Phys. Rev. **D84** (2011) 024025; Phys. Rev. **D88** (2013) 084014; Phys. Rev. **D90** (2014) 124068; Phys. Rev. **D94** (2016) 044019] extended to arbitrary strong and time-dependent gravitational, inertial *and* electromagnetic fields
- Exact Foldy-Wouthuysen transformation constructed
- Quantum and semiclassical equations of motion agree with classical Mathisson-Papapetrou theory of spin
- Possible applications include analysis of spin dynamics in (weak and strong) gravitational wave – new detectors?

Thanks !