Quantum corrections to the Classical Statistical Approximation

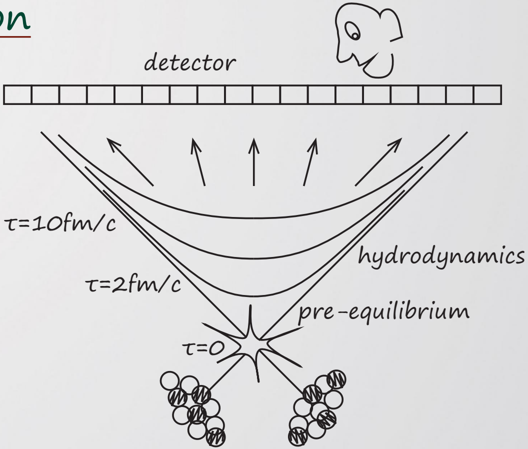
A. V. Leonidov, A. A. Radovskaya

I.E.Tamm Department of Theoretical Physics, P.N.Lebedev Physics Institute, 119991 Moscow, Russia

Plan of the talk

- Motivation
- Keldysh technique and general formalism
- Toy models and numerical results
- Conclusions

Motivation



The Little Bang

- Nucl.Phys. A850 (2011) 69-109; Explanation of the fast izotropisation in HIC for the scalar field. CSA for the smal coupling constant only.
- CSA physically natural: HIC, early Universe, cold Bose gases, chemical physics...
- What can be done beyond the CSA??

Keldysh technique

The main problem under consideration is to build up the description of the field relaxation from the highly excited initial state:

- The initial state of the system is not the vacuum.
- Nonequilibrium evolution from t_0 to t_1.
- We need specify the initial state.

Keldysh approach

– Describe the time evolution of the density matrix

$$\hat{\rho}(t) = \sum_i P_i \, |\psi_i(t)\rangle\langle\psi_i(t)|$$

$$\hat{\rho}(t) = \hat{U}(t, t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t, t_0)$$

– An observable is given by

$$\langle A(t)\rangle = \mathrm{tr}(\hat{A}\hat{\rho}) =$$

$$= \mathrm{tr}(\hat{U}^\dagger(t, t_0)\ \hat{A}\ \hat{U}(t, t_0)\ \hat{\rho}(t_0))$$

Keldysh technique: general

Suppose we know the density matrix $\hat{\rho}(t)$ at the initial time t_0 and want to calculate the observable $F(\varphi)$ at the moment t_1

$$\langle F(\hat{\varphi})\rangle_{t_1} = tr(F(\hat{\varphi})\hat{\rho}(t_1)) = \int \mathfrak{D}\xi(\vec{x})\ F(\xi)\langle\xi|\hat{U}(t_1,t_0)\hat{\rho}(t_0)\hat{U}(t_0,t_1)|\xi\rangle$$

$$= \int \mathfrak{D}\xi \int \mathfrak{D}\xi_1 \int \mathfrak{D}\xi_2\ F(\xi)\langle\xi|\hat{U}(t_1,t_0)|\xi_1\rangle\langle\xi_1|\hat{\rho}(t_0)|\xi_2\rangle\langle\xi_2|\hat{U}(t_0,t_1)|\xi\rangle.$$

Here $|\xi\rangle$ is an eigenstate of the field operator $\hat{\varphi}(\vec{x})|\xi\rangle = \xi(\vec{x})|\xi\rangle$ and $\int \mathfrak{D}\xi(\vec{x})$ is a path integral over all possible 3-d functions originating from unity operator $\hat{1} = \int \mathfrak{D}\xi(\vec{x})\ |\xi\rangle\langle\xi|$.

The matrix elements of the evolution operator are the path integrals over 4-d functions $\mathcal{D}\eta(,\vec{x})$

$$\langle\xi|\hat{U}(t_1,t_0)|\xi_1\rangle = \int_{\eta_F(t_0,\vec{x})=\xi_1(\vec{x})}^{\eta_F(t_1,\vec{x})=\xi(\vec{x})} \mathcal{D}\eta_F(t,\vec{x})e^{iS[\eta_F]},$$

$$\langle\xi_2|\hat{U}(t_0,t_1)|\xi\rangle = \int_{\eta_B(t_0,\vec{x})=\xi_2(\vec{x})}^{\eta_B(t_1,\vec{x})=\xi(\vec{x})} \mathcal{D}\eta_B(t,\vec{x})e^{-iS[\eta_B]}$$

Keldysh technique: general

The exptectation value of observable $F(\hat{\varphi})$ through the path integral

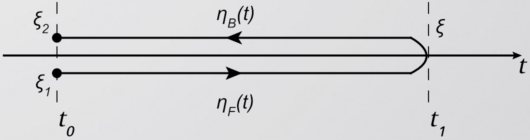
$$\langle F(\hat{\varphi})\rangle_{t_1} = \int \mathfrak{D}\xi \int \mathfrak{D}\xi_1 \int \mathfrak{D}\xi_2 \; \langle \xi_1 | \hat{\rho}(t_0) | \xi_2 \rangle$$

$$\times F(\xi) \int_{\eta_F(t_0,\vec{x})=\xi_1(\vec{x})}^{\eta_F(t_1,\vec{x})=\xi(\vec{x})} \mathcal{D}\eta_F(t,\vec{x}) \int_{\eta_B(t_0,\vec{x})=\xi_2(\vec{x})}^{\eta_B(t_1,\vec{x})=\xi(\vec{x})} \mathcal{D}\eta_B(t,\vec{x}) \; e^{iS[\eta_F]-iS[\eta_B]}.$$

The Keldysh action is defined as $S_K[\eta_F, \eta_B] = S[\eta_F] - S[\eta_B]$

-Time flow from t_0 to t_1 and backward.

-The fields η_F and η_B live on Keldysh contour:



Keldysh technique: general

Let us change the variables to

$$\phi_c = \frac{\eta_F + \eta_B}{2}, \qquad \phi_q = \eta_F - \eta_B.$$

Here ϕ_c is so-called "classical" component and ϕ_q is "quantum" component. After some simple algebra we, finally, have

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int \mathfrak{D}\chi_1 \int \mathfrak{D}\xi_1 \int \mathfrak{D}\xi_2 \; \langle \xi_1 | \hat{\rho}(t_0) | \xi_2 \rangle$$

$$\times \int_{\phi_c(t_0,\vec{x}) = \frac{\xi_1(\vec{x}) + \xi_2(\vec{x})}{2}}^{\phi_c(\infty,\vec{x}) = \chi_1(\vec{x})} \mathcal{D}\phi_c \int_{\phi_q(t_0,\vec{x}) = \xi_1(\vec{x}) - \xi_2(\vec{x})}^{\phi_q(\infty,\vec{x}) = 0} \mathcal{D}\phi_q \; F(\phi_c(t_1)) \; e^{iS_K[\phi_c,\phi_q]}$$

Keldysh technique: φ^4

We use φ^4 theory for simplicity

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{g^2}{4}\varphi^4 + J\varphi$$

The Keldysh action

$$S_K[\phi_c,\phi_q] = \int d^3x\ \dot{\phi}_c(t_0,\vec{x})(\xi_1(\vec{x}) - \xi_2(\vec{x}))$$

$$- \int_{t_0}^{\infty} dt \int d^3x \Big[\phi_q\big(\underbrace{\partial_\mu\partial^\mu\phi_c + g^2\phi_c^3 - J}_{\text{equation of motion}}\big) - \frac{g^2}{4}\phi_c\phi_q^3\Big]$$

What to do next?

- remember about Plank constant
- substitute $\phi_q \to \hbar\phi_q$
- use semiclassical decomposition

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The Keldysh action

$$\frac{1}{\hbar}S_K[\phi_c,\phi_q] = \frac{1}{\hbar}\int d^3x\ \dot{\phi}_c(t_0,\vec{x})(\xi_1(\vec{x}) - \xi_2(\vec{x}))$$

$$-\frac{1}{\hbar}\int_{t_0}^{\infty} dt \int d^3x\left[\phi_q\big(\underbrace{\partial_\mu\partial^\mu\phi_c + g^2\phi_c^3 - J}_{\text{equation of motion}}\big) - \frac{g^2}{4}\phi_c\phi_q^3\right]$$

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Leading Order

Semiclassical decomposition

$$e^{-i\frac{g^2\hbar^2}{4}\int_{t_0}^{\infty} dt \int d^3x\ \phi_c\phi_q^3} = \underbrace{1}_{LO} - \underbrace{\frac{ig^2\hbar^2}{4}\int_{t_0}^{\infty} dt \int d^3x\ \phi_c\phi_q^3}_{NLO} + \ldots$$

At Leading Order we have

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int \mathfrak{D}\chi_1 \int \mathfrak{D}\xi_1 \int \mathfrak{D}\xi_2\ \langle \xi_1|\hat{\rho}(t_0)|\xi_2\rangle$$

$$\times \int_{\phi_c(t_0,\vec{x})=\frac{\xi_1(\vec{x})+\xi_2(\vec{x})}{2}}^{\phi_c(\infty,\vec{x})=\chi_1(\vec{x})} \mathcal{D}\phi_c e^{i\int d^3x\ \dot{\phi}_c(t_0,\vec{x})(\xi_1(\vec{x})-\xi_2(\vec{x}))} \int_{\phi_q(t_0,\vec{x})=\xi_1(\vec{x})-\xi_2(\vec{x})}^{\phi_q(\infty,\vec{x})=0} \mathcal{D}\phi_q\ F(\phi_c(t_1))$$

$$\times e^{i\int dtd^3x\ \phi_q(\partial_\mu\partial^\mu\phi_c + g^2\phi_c^3 - J)}$$

- perform integration over field ϕ_q to receive functional delta function from equation of motion
- insert "initial velocity" unity $1 = \int \mathfrak{D}p(\vec{x})\delta(\tilde{p}(\vec{x}) - \dot{\phi}_c(t_0,\vec{x}))$
- perfrom integration over ϕ_c with help of EoM and boundary conditions
- change variables to $\alpha = \frac{\xi_1+\xi_2}{2}$, $\beta = \xi_1 - \xi_2$

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At Leading Order we have

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int \mathfrak{D}\alpha(\vec{x})\mathfrak{D}p(\vec{x}) f_W[\alpha(\vec{x}), p(\vec{x})), t_0] F(\phi_{cl}(t_1))$$

$$f_W[\alpha(\vec{x}), p(\vec{x}), t_0] = \int \mathfrak{D}\beta(\vec{x})\langle\alpha + \frac{\beta}{2}|\hat{\rho}(t_0)|\alpha - \frac{\beta}{2}\rangle e^{i\int d^3x \; p(\vec{x})\beta(\vec{x})}$$

$$\partial_\mu\partial^\mu\phi_{cl} + g^2\phi_{cl}^3 = 0$$

$$\phi_{cl}(t_0, \vec{x}) = \alpha(\vec{x})$$

$$\dot{\phi}_{cl}(t_0, \vec{x}) = p(\vec{x})$$

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Classical Statistical Approximation

Leading order recipe is
- find solution of the classical EoM
- calculate observable on this solution
- average over all initial conditions with weight of the Wigner functional

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int \mathfrak{D}\alpha(\vec{x})\mathfrak{D}p(\vec{x}) f_W[\alpha(\vec{x}), p(\vec{x})), t_0] F(\phi_{cl}(t_1))$$

$$f_W[\alpha(\vec{x}), p(\vec{x}), t_0] = \int \mathfrak{D}\beta(\vec{x}) \langle \alpha + \frac{\beta}{2} | \hat{\rho}(t_0) | \alpha - \frac{\beta}{2} \rangle e^{i \int d^3x \; p(\vec{x})\beta(\vec{x})}$$

$$\partial_\mu \partial^\mu \phi_{cl} + g^2 \phi_{cl}^3 = 0$$

$$\phi_{cl}(t_0, \vec{x}) = \alpha(\vec{x})$$

$$\dot{\phi}_{cl}(t_0, \vec{x}) = p(\vec{x})$$

- We do not need the small coupling constant for CSA.
- When the CSA does not work?
- We need NLO answers !

Notation

We introduce notation for averaging over initial condition with the Wigner functional as

$$\langle \mathcal{O} \rangle_{i.c.} = \int \mathfrak{D}\alpha(\vec{x})\mathfrak{D}p(\vec{x}) f_W[\alpha(\vec{x}), p(\vec{x})), t_0]\ \mathcal{O}$$

So, the LO answer can be written simply as

$$\langle F(\hat{\varphi}) \rangle_{t_1}^{LO} = \langle\ F(\phi_{cl}(t_1))\ \rangle_{i.c.}$$

Next-to-Leading Order

$$e^{-i\frac{g^2}{4}\int_{t_0}^{\infty} dt \int d^3x \; \phi_c \phi_q^3} = \underbrace{1}_{LO} - \underbrace{\frac{ig^2}{4}\int_{t_0}^{\infty} dt \int d^3x \; \phi_c \phi_q^3}_{NLO} + \ldots$$

Due to $\phi_q J$ term in the action we can rewrite each ϕ_q field as functional derivative

$$\frac{\delta}{\delta J(t', \vec{x}')} e^{iS_K[\phi_c, \phi_q]} = -i\phi_q(t', \vec{x}') e^{iS_K[\phi_c, \phi_q]}$$

And NLO answer can be obtained as

$$\langle F(\hat{\varphi})\rangle_{t_1}^{NLO} = \frac{g^2}{4}\left\langle \int_{t_0}^{t_1} dt' \int d^3x' \phi_{cl}(t', \vec{x}') \frac{\delta^3 F(\phi_{cl}(t_1, \vec{x}))}{\delta J^3(t', \vec{x}')}\bigg|_{J=0}\right\rangle_{i.c.}$$

And LO + NLO solution is

$$\langle F(\hat{\varphi})\rangle_{t_1}^{LO+NLO} = \left\langle F(\phi_{cl}(t_1, \vec{x})) + \frac{g^2}{4}\int_{t_0}^{t_1} dt' \int d^3x' \phi_{cl}(t', \vec{x}') \frac{\delta^3 F(\phi_{cl}(t_1, \vec{x}))}{\delta J^3(t', \vec{x}')}\bigg|_{J=0}\right\rangle_{i.c.}$$

Next-to-Leading Order

Let us define k-th variation of the classical solution over source J as

$$\frac{\delta^k \phi_{cl}(t_1,\vec{x}_1)}{\delta J^k(t_2,\vec{x}_2)} = \Phi_k(t_1,\vec{x}_1;t_2,\vec{x}_2).$$

Then

$$\frac{\delta^3 F(\phi_{cl}(t_1,\vec{x}_1))}{\delta J^3(t_2,\vec{x}_2)} = \frac{\partial F}{\partial \phi_{cl}}\Phi_3 + 3\frac{\partial^2 F}{\partial \phi_{cl}^2}\Phi_1\Phi_2 + \frac{\partial^3 F}{\partial \phi_{cl}^3}\Phi_1^3.$$

Variations $\Phi_k(t_1,\vec{x}_1;t_2,\vec{x}_2)$ can be found by variation of the classical EoM

$$\frac{\delta^3}{\delta J^3(t_2,\vec{x}_2)}\left(\partial_\mu\partial^\mu\phi_{cl}(t_1,\vec{x}_1) + g^2\phi_{cl}^3(t_1,\vec{x}_1) = J(t_1,\vec{x}_1)\right),$$

that gives

$$L_{t_1}\Phi_1(t_1,\vec{x}_1;t_2,\vec{x}_2) = \delta(t_1-t_2)\delta^{(3)}(\vec{x}_1-\vec{x}_2)$$

$$L_{t_1}\Phi_2(t_1,\vec{x}_1;t_2,\vec{x}_2) = -6g^2\phi_{cl}(t_1,\vec{x}_1)\Phi_1^2(t_1,\vec{x}_1;t_2,\vec{x}_2)$$

$$L_{t_1}\Phi_3(t_1,\vec{x}_1;t_2,\vec{x}_2) = -6g^2\Phi_1^3(t_1,\vec{x}_1;t_2,\vec{x}_2) - 18g^2\phi_{cl}(t_1,\vec{x}_1)\Phi_1(t_1,\vec{x}_1;t_2,\vec{x}_2)\Phi_2(t_1,\vec{x}_1;t_2,\vec{x}_2)$$

$$L_{t_1} = \partial_{t_1}^2 - \partial_{\vec{x}_1}^2 + 3g^2\phi_{cl}^2(t_1,\vec{x}_1)$$

Quantum correction s to the CSA: recipe

$$\langle F(\hat{\varphi})\rangle_{t_1}^{LO+NLO} = \left\langle F(\phi_{cl}(t_1,\vec{x})) + \frac{g^2}{4}\int_{t_0}^{t_1} dt' \int d^3x'\,\phi_{cl}(t',\vec{x}')\,\frac{\delta^3 F(\phi_{cl}(t_1,\vec{x}))}{\delta J^3(t',\vec{x}')}\Big|_{J=0}\right\rangle_{i.c.}$$

$$\frac{\delta^3 F(\phi_{cl}(t_1,\vec{x}_1))}{\delta J^3(t_2,\vec{x}_2)} = \frac{\partial F}{\partial\phi_{cl}}\Phi_3 + 3\frac{\partial^2 F}{\partial\phi_{cl}^2}\Phi_1\Phi_2 + \frac{\partial^3 F}{\partial\phi_{cl}^3}\Phi_1^3.$$

Variations

$$L_{t_1}\Phi_1(t_1,\vec{x}_1;t_2,\vec{x}_2) = \delta(t_1-t_2)\delta^{(3)}(\vec{x}_1-\vec{x}_2)$$

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$$L_{t_1} = \partial_{t_1}^2 - \partial_{\vec{x}_1}^2 + 3g^2\phi_{cl}^2(t_1,\vec{x}_1)$$

Averaging over the initial conditions

$$f_W[\alpha(\vec{x}),p(\vec{x}),t_0] = \int \mathfrak{D}\beta(\vec{x})\langle\alpha+\frac{\beta}{2}|\hat{\rho}(t_0)|\alpha-\frac{\beta}{2}\rangle e^{i\int d^3x\, p(\vec{x})\beta(\vec{x})}$$

$$\partial_\mu\partial^\mu\phi_{cl} + g^2\phi_{cl}^3 = 0$$

$$\phi_{cl}(t_0,\vec{x}) = \alpha(\vec{x})$$

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Toy model: Spatially homogeneous static box

Spatially homogeneous case $\partial_i \varphi(t, x) = 0$ LO and NLO terms can be found analytically. (almost)

$$S = V \int dt \left(\frac{1}{2}\dot{\varphi}^2 - \frac{g^2}{4}\varphi^4 + J\varphi \right), \qquad V = \int d^3x$$

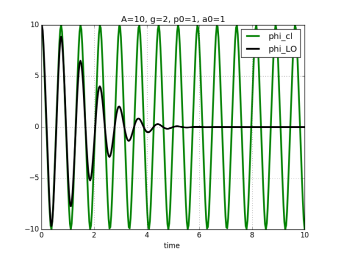
The equation of motion is

$$\ddot{\varphi} + g^2\varphi^3 = J.$$

And its solution for J=0 is periodic Jacobi elliptic function with period T_{cl}:

$$\phi_{cl}(t) = \phi_m cn\left(\frac{1}{2}, g\phi_m t + C\right),$$

$$T_{cl} = \frac{4}{g\phi_m}K(1/2), \qquad K(1/2) \approx 1.85.$$



CSA at work (numerical)

$$\langle \hat{\varphi} \rangle^{LO}_{t_1} = \left\langle \phi_{cl}(t_1) \right\rangle_{i.c.}$$

$$\equiv \int d\alpha \int dp\, f_W(\alpha, p)\phi_{cl}(t_1),$$

$$f_W(\alpha, p) = \frac{1}{\pi\alpha_0 p_0} e^{-\frac{(\alpha - A)^2}{\alpha_0^2}} e^{-\frac{p^2}{p_0^2}}.$$

T^{μ}_{μ} full

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\varphi\partial^\lambda\varphi - \frac{g^2}{4}\varphi^4\right)$$

$$\varepsilon = T^{00} = \frac{1}{2}\dot{\varphi}^2 + \frac{g^2}{4}\varphi^4$$

$$p = T^{ii} = \frac{1}{2}\dot{\varphi}^2 - \frac{g^2}{4}\varphi^4$$

For hydrodynamic to start we need equation of state $\langle T^{\mu}_{\mu}\rangle = \varepsilon - 3p = 0$

On classical level

$$T^{\mu}_{\mu} = \varepsilon - 3p = -\dot{\varphi} + g^2\varphi^4.$$

One can calculate this observable on the full solution as

$$\langle T^{\mu}_{\mu}\rangle_{t_1} = \int d\xi\ [\xi_1, \rho_0, \xi_2] \int\limits_{\varphi_c(t_0)=\frac{\xi_1+\xi_2}{2}}^{\varphi_c(\infty)=0} \mathcal{D}\varphi_c \int\limits_{\varphi_q(t_0)=\xi_1-\xi_2}^{\varphi_q(\infty)=0} \mathcal{D}\varphi_q\ e^{iS_K[\varphi_c,\varphi_q]}\left(-\dot{\varphi}_c^2 + g^2\varphi_c^4\right).$$

T^{μ}_{μ} full

Consider variation of the Keldysh action over the field φ_q

$$\frac{\delta S_K}{\delta \varphi_q}\Big|_{J=0} = -V_3\left(\ddot{\varphi}_c + g^2\varphi_c^3 + \frac{3}{4}g^2\varphi_c\varphi_q^2\right).$$

We can use φ_c^3 term as

$$\langle T^{\mu}_{\mu}\rangle_{t_1} = \int d\xi\ [\xi_1,\rho_0,\xi_2] \int \mathcal{D}\varphi_c\mathcal{D}\varphi_q\Big(-\dot{\varphi}_c^2 - \frac{\varphi_c}{V_3}\frac{\delta S_K}{\delta\varphi_q} - \varphi_c\ddot{\varphi}_c - \frac{3}{4}g^2\varphi_c^2\varphi_q^2\Big)e^{iS_K[\varphi_c,\varphi_q]}.$$

The red terms are zero

- Evaluate it by parts and neglect the surface term

$$\frac{1}{V_3}\int \mathcal{D}\varphi_c\mathcal{D}\varphi_q\cdot\varphi_c\frac{\delta S_K}{\delta\varphi_q}e^{iS_K[\varphi_c,\varphi_q]} = \frac{1}{V_3}\int \mathcal{D}\varphi_c\mathcal{D}\varphi_q\cdot\varphi_c\frac{\delta}{\delta\varphi_q}\left(e^{iS_K[\varphi_c,\varphi_q]}\right) = 0$$

- Rotate the fields back

$$\varphi_q(t) = \eta_F(t) - \eta_B(t),$$

$$\int \mathcal{D}\eta_F\mathcal{D}\eta_B\ \eta_F(t_1)e^{iS_K[\eta_F,\eta_B]} = \int \mathcal{D}\eta_F\mathcal{D}\eta_B\ \eta_B(t_1)e^{iS_K[\eta_F,\eta_B]}.$$

T^μ_μ full

Two remaining terms can be expressed through the total time derivative as

$$\langle T^\mu_\mu \rangle_{t_1} = -\frac{1}{2}\int d\xi\ [\xi_1, \rho_0, \xi_2] \int \mathcal{D}\varphi_c \mathcal{D}\varphi_q\ e^{iS_K[\varphi_c,\varphi_q]}\ \partial^2_{t_1} \varphi^2_c(t_1).$$

Physically, the field in the static box will relaxate to some constant with time. It means, that the trace $\langle T^\mu_\mu \rangle_{t_1\to\infty} = 0$ for the full theory (at all orders in semiclassical decomposition)

However, we can check directly it for LO and NLO as

$$\langle T^\mu_\mu \rangle^{LO+NLO}_{t_1} = -\frac{1}{2}\int d\alpha dp\ f_W(\alpha, p, t_0)\ \partial^2_{t_1}\Big(\phi^2_{cl}(t_1)+$$

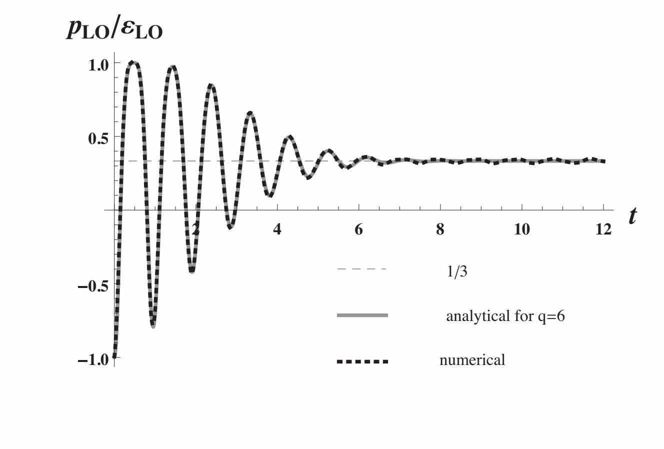
$$\frac{1}{2V^2 g^2 \phi_m^4}\int_{z_0}^{z_1} dz_2 f_0(z_2)(f_0(z_1)f_3(z_1,z_2) + 3f_1(z_1,z_2)f_2(z_1,z_2))\Big)$$

$$= -\frac{1}{2}\int d\alpha dp\ f_W(\alpha, p, t_0)\ \partial^2_{t_1}\Big(\phi^2_m f^2_0(g\phi_m t_1)+$$

$$\frac{1}{2V^2 g^2 \phi_m^4}\Big[\psi_0(g\phi_m t_1) + g\phi_m \psi_1(g\phi_m t_1)t_1 + g^2\phi^2_m\psi_2(g\phi_m t_1)t_1^2 + g^3\phi^3_m\psi_3(g\phi_m t_1)t_1^3\Big]\Big).$$

T^{μ}_{μ} LO

$$\frac{p_{LO}(t_1 \to \infty)}{\varepsilon_{LO}} = \left[\frac{1}{3} + 8\,\mathcal{I}(2) e^{-\frac{4\pi^2 p_0^2}{g^2 A^4 T^2}} e^{-\frac{4\alpha_0^2 \pi^2 g^2}{T^2} t_1^2} \cos\left(\frac{4\pi A}{T} g t_1 \right) + \dots \right]$$



T^{μ}_{μ} NLO

Fourier decomposition of the NLO periodic functions

$$\psi_n(t+T_{cl}) = \psi_n(t), \qquad n = 0, 1, 2, 3,$$

$$\psi_n(t) = \sum_{k=-\infty}^{\infty} \psi_n^{(k)} e^{ikt\frac{2\pi}{T_{cl}}}.$$

The integration over initial conditions with "good " Wigner function gives

$$\int d\alpha dp\, f_W(\alpha, p) \sum_{k=-\infty}^{\infty} \psi_n^{(k)} e^{ikt\frac{2\pi}{T_{cl}}} = \sum_{k=-\infty}^{\infty} \psi_n^{(k)} A e^{-Bk^2 t^2} e^{iCkt}$$

The only dangerous term for the trace is $k = 0$

$$\langle T^{\mu}_{\mu} \rangle_{t_1}^{LO+NLO} \approx \psi_n^{(0)}\ t^n\ const$$

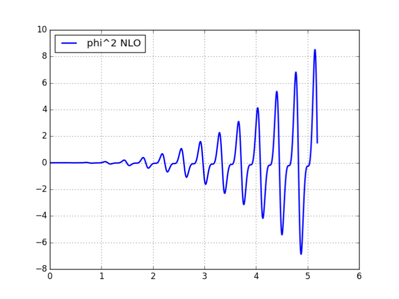
$$F(z_1) = \psi_0(z_1) + \psi_1(z_1) z_1 + \psi_2(z_1) z_1^2 + \psi_3(z_1) z_1^3.$$

T^{μ}_{μ} NLO

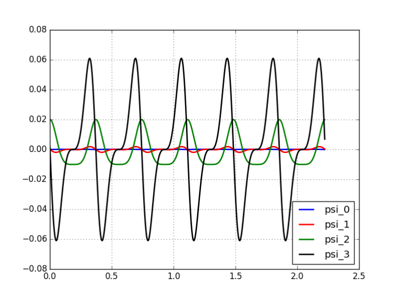
The Vandermonde matrix

$$\begin{pmatrix} F(z) \\ F(z+T) \\ F(z+2T) \\ F(z+3T) \end{pmatrix} = \begin{pmatrix} 1 & z & z^2 & z^3 \\ 1 & z+T & (z+T)^2 & (z+T)^3 \\ 1 & z+2T & (z+2T)^2 & (z+2T)^3 \\ 1 & z+3T & (z+3T)^2 & (z+3T)^3 \end{pmatrix} \begin{pmatrix} \psi_0(z) \\ \psi_1(z) \\ \psi_2(z) \\ \psi_3(z) \end{pmatrix}$$

$\varphi^2(t)^{NLO}$



$\psi_n(t)$



Longitudinally expanding box

$$\tau^2 = t^2 - z^2,$$
$$\eta = \frac{1}{2}\ln\frac{t+z}{t-z},$$

In ''homogeneous'' case $\partial_\eta \varphi = 0$ and $\partial_\perp \varphi = 0$

$$S = V_2 \int d\tau\ \tau\Big(\frac{1}{2}\dot{\varphi}^2 - \frac{g^2}{4}\varphi^4 + J\varphi\Big)$$
$$V_2 = \int d^2 x_\perp d\eta$$

Equation of motion can not be calculated analytically

$$\partial_\tau^2 \varphi + \frac{1}{\tau}\partial_\tau \varphi + g^2 \varphi^3 = J$$

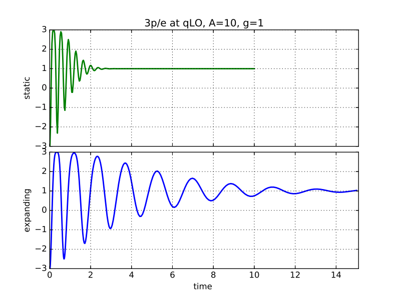
Change of variables lead to almost periodical solution

$$y = \tau^{\frac{2}{3}},$$
$$\varphi(\tau) = \tau^{-\frac{1}{3}} \xi(\tau^{\frac{2}{3}}),$$
$$\ddot{\xi}(y) + \frac{1}{4y^2}\xi(y) + \frac{9}{4} g^2 \xi(y)^3 = 0,$$
$$\xi(y) = \xi_m cn(\bar{g}\xi_m y + C), \qquad \bar{g} = \frac{3}{2} g$$

Static vs expanding box

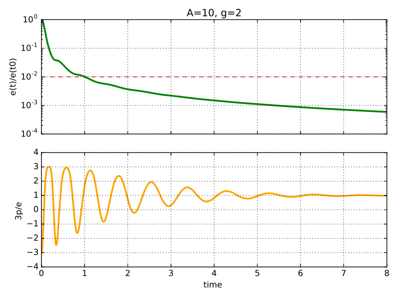
T^{μ}_{μ} at LO for static and expanding cases

Static vs. Expanding system



- termalization time larger in expanding system with identical initial conditions

Energy loss due to expansion



- there is only 1% energy density remains, however the system does not termalized ($\varepsilon \neq 3p$)

T^{μ}_{μ} full expanding

$$\langle T^{\mu}_{\mu}\rangle_{\tau_1} = -\frac{1}{2}\int d\xi\ [\xi_1,\rho_0,\xi_2]\int \mathcal{D}\varphi_c\mathcal{D}\varphi_q\ e^{iS_K^{exp}[\varphi_c,\varphi_q]}\ \left(\partial^2_{\tau_1}+\frac{1}{\tau_1}\partial_{\tau_1}\right)\varphi_c^2(\tau_1)$$

The system is expanding:

- Asymptotically T^{μ}_{μ} but when $\varepsilon = 0$ and $p = o$ due to expansion
- We are looking for intermediate quasi stationary state with definite EoS
- We need to take into account expansion, hence $T^{\mu}_{\mu}/\varepsilon_{LO}$, where $\varepsilon_{LO} \approx \tau^{-4/3}$

$$\frac{1}{\varepsilon_{LO}}\langle T^{\mu}_{\mu}\rangle^{LO+NLO}_{\tau_1} = -\frac{\tau_1^{\frac{4}{3}}}{2\varepsilon^0_{LO}}\int d\alpha dp\ f_W(\alpha,p,t_0)\ (\partial^2_{\tau_1}+\frac{1}{\tau_1}\partial_{\tau_1})\frac{\tau_1^{-\frac{2}{3}}}{2V_2^2\bar{g}^2\xi_m^4}\left(\frac{3}{2}\right)^2$$
$$\times\left[\psi_0(\bar{g}\xi_m\tau_1^{\frac{2}{3}})+\bar{g}\xi_m\psi_1(\bar{g}\xi_m\tau_1^{\frac{2}{3}})\tau_1^{\frac{2}{3}}+\bar{g}^2\xi_m^2\psi_2(\bar{g}\xi_m\tau_1^{\frac{2}{3}})\tau_1^{\frac{4}{3}}+\bar{g}^3\xi_m^3\psi_3(\bar{g}\xi_m\tau_1^{\frac{2}{3}})\tau_1^2\right].$$

T^{μ}_{μ} full expanding

$$\frac{1}{\varepsilon_{LO}}\langle T^{\mu}_{\mu}\rangle^{LO+NLO}_{\tau_1} = \langle k_1(g,\xi_m)t^{-n} + k_2(g^0,\xi_m)t^0 + k_1(g,\xi_m)t^m, \quad m \le 2\rangle_{i.c.}$$

The form of the Wigner function controls the value of the NLO correction.

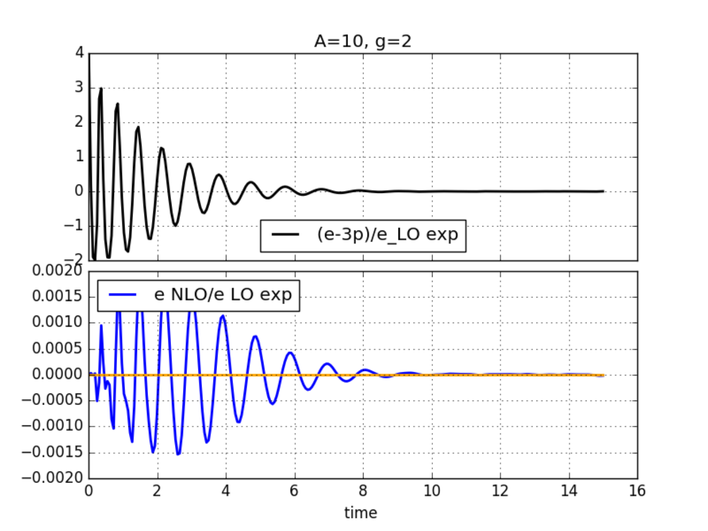
$$f_W(\alpha,p) = \frac{1}{\pi\alpha_0 p_0} e^{-\frac{(\alpha-A)^2}{\alpha_0^2}} e^{-\frac{p^2}{p_0^2}}$$

The parameter A mimic the measure of the field excitation. Then larger A, then better CSA works.

$$\ddot{\xi}(y) + \frac{1}{4y^2}\xi(y) + \frac{9}{4}g^2\xi(y)^3 = 0$$

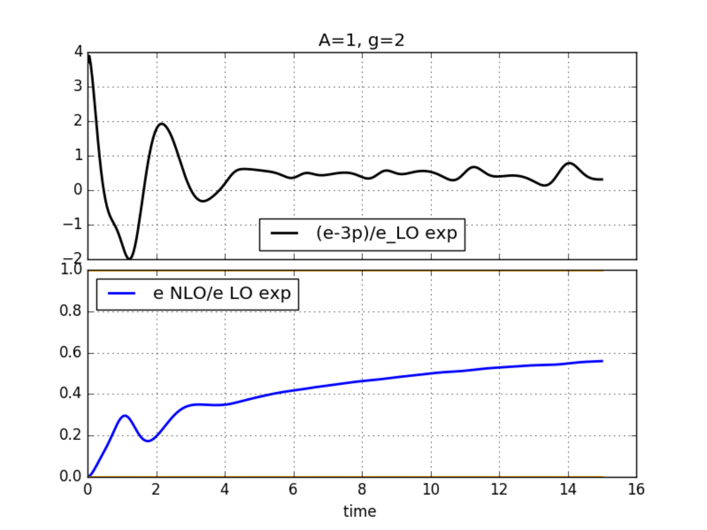
Numerical results

$A = 10$, $g = 2$. CSA works excellent. $\varepsilon - 3p = 0$



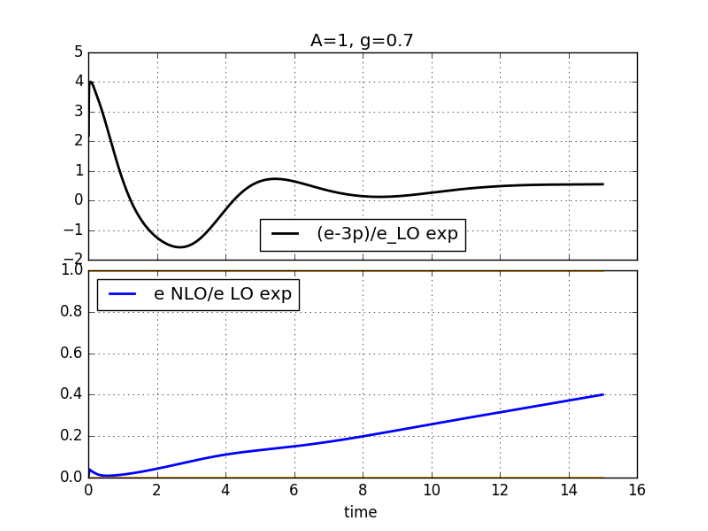
Numerical results

A $= 1$, $g = 2$. CSA works fine. $\varepsilon - 3p =$ const ≈ 0.5



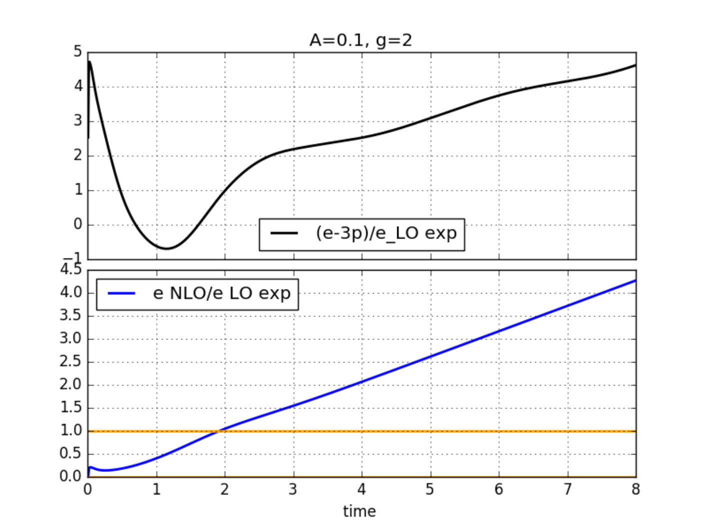
Numerical results

$A = 1$, $g = 0.7$. CSA works fine. $\varepsilon - 3p = const \approx 0.5$



Numerical results

$A = 0.1,\ g = 2.$ CSA does not work



Conclusions

- The systematic procedure for calculation of quantum corrections to the Classical Statistical Approximation was developed.
- Time evolution of the $\langle T^\mu_\mu \rangle$ was analyzed for homogeneous static and longitudinally expanding models.
- It was shown that quantum corrections can change the CSA predictions.

Thank you for your attention

:-)