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# Nonlinear Schrödinger equation and classical-field description of quantum phenomena

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# Quantum theory vs Classical theories

**Orthodox point of view (Copenhagen interpretation):** In consistent quantum theory, the atoms and radiation are quantized. This results in a paradox: the wave-particle duality.

**Semiclassical approach:** the radiation field is treated as a classical field while the atoms are quantized (Lamb W.E., Scully M.O., Jaynes E.T. , L. Mandel, Barut A.O., Stroud C.R. Jr., Crisp M.D., Nesbet R.K. et al.).

**Purely classical theories:** operates only by classical (continuous in space and time) fields or/and by classical particles. All physical objects are either the fields or the particles.

I will show that quantum mechanics can be constructed as a classical field theory without using such notions as particles (photons and electrons.

I will show that quantum mechanics can be considered as not a theory of particles but a classical field theory in the spirit of classical electrodynamics.

# Myths of quantum mechanics

Quantum mythology is based on some basic physical effects that, as considered, cannot be explained within the framework of classical ideas and require quantization.

These effects include:

- thermal radiation;
- photoelectric effect;
- Compton effect;
- Born rule;
- Heisenberg uncertainty principle;
- stability of atoms
- discrete emission spectra of atoms;
- spontaneous “transitions” and necessity of zero-point fluctuations;
- light-atom interactions;
- discrete events in the light-detector interaction, (e.g. the double-slit experiments);
- electrons diffraction and double-slit experiments with electrons;
- spin;
- Hanbury Brown and Twiss experiments;
- Lamb shift;
- anomalous magnetic moment of electron;
- Stern and Gerlach experiment;
- violation of the Bell inequalities;
- EPR and EPRB experiments.

# Electron

Pauli equation can be written in hydrodynamic form (T. Takabayasi, 1954).

From a formal perspective, these equations describe the flow of an electrically charged magnetic fluid in an external electromagnetic field.

Thus the object which is described by the Pauli equation (i.e. the “electron”) can be considered as a real field (continuous medium) which possesses the electric charge, electric current, internal angular momentum and internal magnetic moment that are continuously distributed in space.

Any elementary volume *dV* of the electron wave has an electric charge $dq = \rho dV$, $\rho = -e\Psi^{\dagger}\Psi$ ;

internal angular momentum $d\mathbf{L}_S = \mathbf{s}dV$, $\mathbf{s} = \frac{\hbar}{2}\Psi^{\dagger}\boldsymbol{\sigma}\Psi$ is the density of internal angular momentum (spin);

and internal magnetic moment $d\boldsymbol{\mu} = \mathfrak{m}dV$, $\mathfrak{m} = \gamma_e \mathbf{s}$ is the density of internal magnetic moment;

$\gamma_e = -\frac{e}{m_e c}$ is the internal gyromagnetic ratio of the electron wave.

For any volume of electron wave $\left|d\mathbf{L}_S\right| = \frac{\hbar}{2}\frac{\left|dq\right|}{e}$.

If some portion of the electron wave has an electric charge equal to $-e$ (e.g. in the hydrogen atom) then it has the internal angular momentum (spin) ½ $\hbar$ and the internal magnetic moment $\mu_B$ (the Bohr magneton).

# What is an electron?

Taking this results into account we can consider an electron wave as a continuous real classical field similar to classical electromagnetic waves.

In this case the wave equations (Dirac, Klein-Gordon, Pauli, Schrödinger) should be considered as the field equations of the electron field similar to Maxwell equations for the classical electromagnetic field.

This perspective corresponds to initial Schrödinger’s point of view.

I will show that this point of view allows describing the basic so-called “quantum” phenomena involving the “electron” within framework of classical field theory.

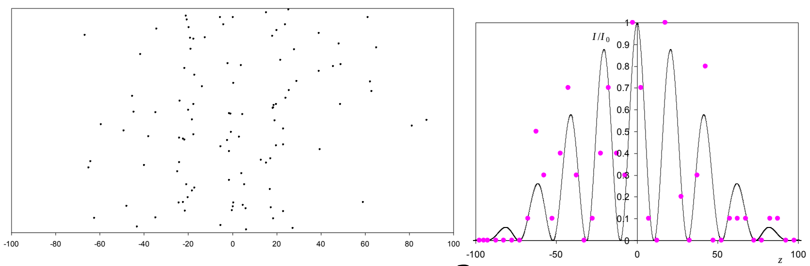
Discrete events in interaction of the “electrons” with a detector can be described as interaction of the continuous electron wave $\psi(\mathbf{r},t)$ with detector if we take into account a discrete (atomic) structure of the detector.

The Born rule can be derived if we consider an interaction of continuous field $\psi(\mathbf{r},t)$ with discrete atoms of a detector which is described by the Schrödinger equation.

The Heisenberg uncertainty principle loses its mystical meaning regarding the fundamental limitations on the accuracy of the measurement of the “electron” parameters and describes the well-known property of the classical wave packet: the more a characteristic width of the wave packet, the smaller the range of the wave numbers in this wave packet and vice versa.

# Double-slit experiment
## (Monte Carlo simulations)

The double-slit experiment is traditionally used for direct demonstration of the wave-particle duality.



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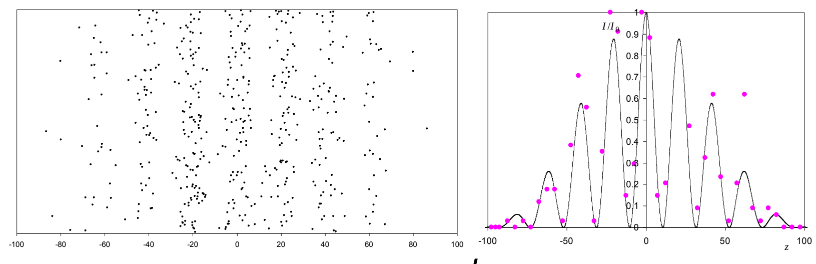

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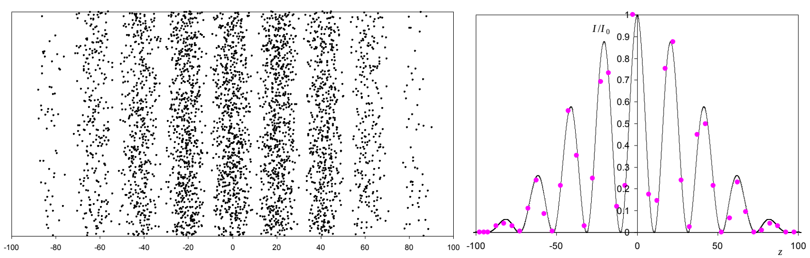

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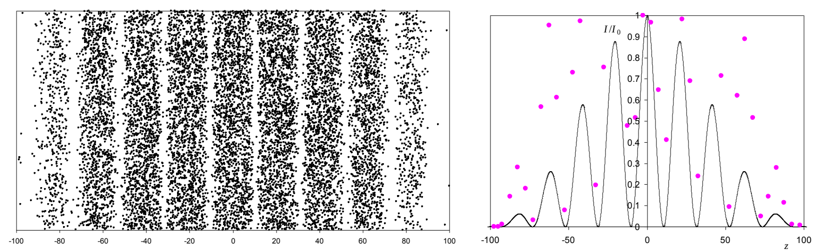

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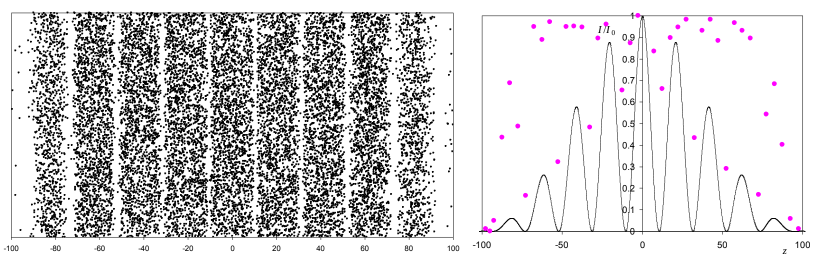

*e*

Interference pattern buildup (left) and the corresponding distribution functions of points on the screen (right) for different exposure times, obtained by Monte Carlo simulation of interaction of electron wave with the detecting screen.

*a)* $\tau$ = 0.02 (100 “electrons”); b) $\tau$ = 0.1 (424 “electrons”); *c)* $\tau$= 1 (3452 “electrons”); d) $\tau$ = 10 (11600 “electrons”); *e)* $\tau$ = 30 (14530 “electrons”).

# Compton Effect

Compton effect can be completely described without using the concept of the particles (photons and electrons) (W. Gordon,1926; P. Dirac,1926; E. Schrödinger,1927; O. Klein and Y. Nishina, 1929)

According to classical electrodynamics, the vector potential of electromagnetic radiation created by a system of electric charges is given by the expression

$$\mathbf{A}_s = \frac{1}{cR_0}\int \mathbf{j}\left(t - R_0/c + (\mathbf{rn})/c\right)dV$$

Electric current density in spinless approximation is

$$\mathbf{j} = i\frac{ec^2}{2\omega_e}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{e^2c}{\hbar\omega_e}\mathbf{A}\Psi\Psi^*$$

where the field Ψ satisfies the Klein-Gordon equation, **A** is the vector potential of planar classical electromagnetic wave.

The term

$$\mathbf{j}_2 = -\frac{e^2c}{\hbar\omega_e}\mathbf{A}\Psi\Psi^*$$

is responsible for the Compton effect.

It is easy to show that the well-known Compton expression

$$\frac{1}{\omega} - \frac{1}{\omega_0} = \frac{\hbar}{m_e c^2}(1 - \cos\vartheta)$$

follows directly from these expressions, while the expressions

$$E + \hbar\omega = E_0 + \hbar\omega_0 \qquad \mathbf{p} + \mathbf{n}\hbar\omega/c = \mathbf{p}_0 + \hbar\mathbf{k}_0$$

are not the laws of conservation of energy and momentum for colliding particles (electron and photon) but the dispersion relation of the system of the fields equations (Maxwell+Klein-Gordon).

# Photoelectric Effect

Three basic laws of the photoelectric effect:
(i) the photoelectric current is proportional to the intensity of incident light;
(ii) the maximum kinetic energy of the emitted photoelectrons varies linearly with the frequency of incident electromagnetic radiation and does not depend on the flux;
(iii) for each substance, there is a threshold frequency (the so-called red edge of the photoelectric effect), below which the photoelectric current is not observed.

The second law contains some interpretation of the experimental facts: it assumes that the electrons are indivisible particles that, at the time of escape from the atom, have a definite kinetic energy.
To rule out any interpretation of the experimental data, we should reformulate the second law: the stopping potential varies linearly with the frequency of the incident electromagnetic radiation and does not depend on the flux.

Considering electron waves as a continuous fields with continuously distributed electric charge, it easy to obtain from Schrödinger equations all three laws of the photoelectric effect:
(i) the mean photoelectric current

$$\overline{I_{ph}} = \beta |\mathbf{E}_0|^2$$

where the parameter $\beta$ does not depend on the incident light intensity $|\mathbf{E}_0|^2$, but it depends on the frequency $\omega_0$ of the incident light.
(ii)

$$U_s = \hbar\omega_0 - \hbar|\omega_1|$$

where $\omega_1$ is the eigenfrequency of the ground state of the atom.
(iii) Analysis shows that $\beta = 0$ for $\omega_0 < |\omega_1|$ and $\beta \approx const$ for $\omega_0 > |\omega_1|$.

# Hydrogen atom

Using the results above, one can say that *the atom represents a classical open volume resonator* in which the charged electron wave - a classical wave field - is held by the electrostatic field of the nucleus.

The electron wave in the hydrogen atom is described by the wave function

$$\psi(t,\mathbf{r}) = \sum_n c_n u_n(\mathbf{r}) \exp(-i\omega_n t)$$

where $c_n$ are the constants that determine the amplitudes of excitation of various eigenmodes of such a resonator; $u_n(\mathbf{r})$ and $\omega_n$ are the eigenfunctions and eigenvalues of the stationary Schrödinger equation.

Because the total charge of the electron wave in the hydrogen atom is equal to $-e$ we have

$$\sum_n |c_n|^2 = 1$$

In the probabilistic interpretation of quantum mechanics, the parameters $|c_n|^2$ are interpreted as the probability of "finding an electron in a quantum state $n$", and this condition is interpreted as a normalization condition for the probabilities.

Considering the electron wave as a classical charged field, we see that the parameters $|c_n|^2$ describe the distribution of the electric charge of the electron wave between the excited eigenmodes of the atom: the value $-e|c_n|^2$ is equal to the electric charge of the electron wave, which is contained in an eigenmode $n$.

# Spontaneous emission

According to classical electrodynamics the intensity of the electric dipole radiation is

$$I = \frac{2}{3c^3}\overline{\ddot{\mathbf{d}}^2}$$

where $\mathbf{d} = -e\int \mathbf{r}|\psi|^2 dV$ is the electric dipole moment of the electron wave in the hydrogen atom. For hydrogen atom is easy to obtain

$$I = \sum_{\omega_{nk}} I_{nk}$$

$$I_{nk} = \frac{4\omega_{nk}^4}{3c^3}|c_n|^2|c_k|^2|\mathbf{d}_{nk}|^2$$

is the intensity of the electric dipole radiation at a frequency of $\omega_{nk} = \omega_n - \omega_k$.
The atom, which is in a pure state does not radiate because in this case, the electron wave has a constant electric dipole moment. That is the pure states are the stationary states of the atom. If an atom is in a mixed state, the electrically charged electron wave has a nonstationary electric dipole moment, and accordingly to classical electrodynamics, it radiates electromagnetic waves at the frequencies $\omega_{nk}$.

Because the hydrogen atom as an open volume resonator has the discrete spectrum of the eigenfrequencies $\omega_n$, the spectrum of the spontaneous emission $\omega_{nk}$ is also discrete.

There are no "jump-like transitions"! *The spontaneous emission of an atom is a continuous process*.

In quantum mechanics, for explanation of the reasons for jump-like transition, the concept of fluctuations of the QED vacuum (Zitterbewegung) is used.
Present theory allows explaining the spontaneous emission without Zitterbewegung: emission occurs continuously and begins immediately after the atom is excited into a mixed excited state.

# Nonlinear Schrödinger equation

The linear Schrödinger equation allows us to describe the source of a spontaneous emission but it cannot describe the change in atom structure that occurs during the spontaneous emission.

Taking into account the inverse action of the spontaneous emission on the electron wave in the framework of classical electrodynamics, one can obtain the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\Delta\psi - e\varphi_p\psi + \frac{2e}{3c^3}\mathbf{r}\dddot{\mathbf{d}}\,\psi$$

where

$$\mathbf{d} = -e\int \mathbf{r}|\psi|^2 dV$$

is the electric dipole moment of the electron wave in the hydrogen atom.
The last term in this equation makes it nonlinear.

It is this nonlinearity of the equation allows describing the redistribution of the electric charge of electron wave during spontaneous emission within the framework of classical field theory which is traditionally considered as electron transitions and described in QED using the concept of Zitterbewegung.

# Light-atom interactions

If an atom is in the field of external electromagnetic wave the nonlinear Schrödinger equation takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2m_e}\Delta\psi - \frac{e^2}{r}\psi + \psi e\mathbf{r}\mathbf{E}_0\cos\omega t - \frac{2e^2}{3c^3}\psi\mathbf{r}\frac{\partial^3}{\partial t^3}\int\mathbf{r}|\psi|^2 d\mathbf{r}$$

For two-level atom we can introduce

$$\rho_{nn} = |c_n|^2, \rho_{kk} = |c_k|^2, \rho_{nk} = c_n c_k^*, \rho_{kn} = c_k c_n^*$$

where

$$\rho_{nn} + \rho_{kk} = 1$$
$$\rho_{nk} = \rho_{kn}^*$$

In this case one can obtain the Optical Bloch equations with dumping due to spontaneous emission

$$\frac{d\rho_{nn}}{dt} = -\frac{d\rho_{kk}}{dt} = i\frac{1}{2}[\rho_{kn}b_{nk}\exp(i\Omega t) - \rho_{nk}b_{nk}^*\exp(-i\Omega t)] - \boldsymbol{2\gamma_{nk}\rho_{nn}\rho_{kk}}$$
$$\frac{d\rho_{nk}}{dt} = \frac{d\rho_{kn}^*}{dt} = (\rho_{kk} - \rho_{nn})[i\frac{1}{2}b_{nk}\exp(i\Omega t) - \gamma_{nk}\rho_{nk}]$$

with correct value of the damping rate

$$\gamma_{nk} = \frac{2\omega_{nk}^3}{3\hbar c^3}|\mathbf{d}_{nk}|^2$$

where

$$b_{nk} = b_{kn}^* = \frac{1}{\hbar}(\mathbf{d}_{nk}\mathbf{E}_0)$$

# Thermal radiation

Nonlinear Schrödinger equation  for two-level atom in an isotropic radiation field can be reduced to optical equation

$$\frac{d\rho_{nn}}{dt} = -\frac{d\rho_{kk}}{dt} = \frac{4\pi}{3\hbar^2}|\mathbf{d}_{nk}|^2 \int_0^\infty \frac{(\rho_{kk}-\rho_{nn})^2\gamma_{nk}}{(\omega-\omega_{nk})^2+(\rho_{kk}-\rho_{nn})^2\gamma_{nk}^2} U_\omega(\omega)d\omega - 2\gamma_{nk}\rho_{nn}\rho_{kk}$$

Knowing the spectral energy density of the radiation field $U_\omega(\omega)$ in which an atom finds itself, one can calculate the parameters $\rho_{nn}$ and $\rho_{kk}$ using equation.

If an atom is in equilibrium with the radiation field when $\frac{d\rho_{nn}}{dt} = \frac{d\rho_{kk}}{dt} = 0$ and

$$\frac{\pi c^3}{\hbar\omega_{nk}^3}\int_0^\infty \frac{(1-2\rho_{nn})^2\gamma_{nk}}{(\omega-\omega_{nk})^2+(1-2\rho_{nn})^2\gamma_{nk}^2} U_\omega(\omega)d\omega = \rho_{nn}(1-\rho_{nn})$$

For continuous spectrum of the radiation field, which satisfies the condition

$$\gamma_{nk}\left|\frac{dU_\omega(\omega_{nk})}{d\omega}\right| \ll U_\omega(\omega_{nk})$$

one obtains

$$\frac{d\rho_{nn}}{dt} = \frac{4\pi^2|\mathbf{d}_{nk}|^2}{3\hbar^2} U_\omega(\omega_{nk})(1-2\rho_{nn}) - 2\gamma_{nk}\rho_{nn}(1-\rho_{nn})$$

If the distribution of the electric charge of the electron wave between the excited eigenmodes of an atom is known *a priori*, then one can find the spectral energy density of radiation at which such a balance is possible:

$$U_\omega(\omega_{nk}) = \frac{\hbar\omega_{nk}^3}{\pi^2 c^3}\frac{\rho_{nn}\rho_{kk}}{\rho_{kk}-\rho_{nn}}$$

# Equilibrium thermal radiation

The spectral energy density of radiation which is in equilibrium with atom is

$$U_\omega(\omega_{nk}) = \frac{\hbar\omega_{nk}^3}{\pi^2 c^3}\frac{\rho_{nn}\rho_{kk}}{\rho_{kk}-\rho_{nn}}$$

for the two-level atom, when only two eigenmodes $k$ and $n$ are excited, we have

$$\rho_{nn} = \frac{\exp\left(-\frac{\hbar\omega_{nk}}{kT}\right)}{1+\exp\left(-\frac{\hbar\omega_{nk}}{kT}\right)}, \qquad \rho_{kk} = \frac{1}{1+\exp\left(-\frac{\hbar\omega_{nk}}{kT}\right)}$$

and the spectral energy density of equilibrium thermal radiation is equal to

$$U_\omega(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\left[\exp\left(\frac{\hbar\omega}{kT}\right)-1\right]\left[1+\exp\left(-\frac{\hbar\omega}{kT}\right)\right]}$$

This expression is different from Planck's law for equilibrium thermal radiation by the additional factor $1/\left[1+\exp\left(-\frac{\hbar\omega}{kT}\right)\right]$.

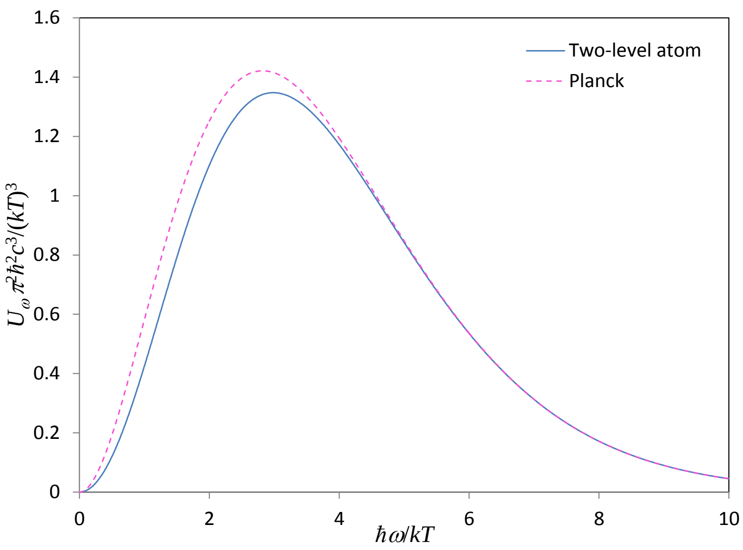


Figure. Dependence of the non-dimensional spectral energy density of equilibrium thermal radiation on the non-dimensional frequency, calculated by using Planck's law and the model of the two-level atom.

# Equilibrium thermal radiation
## (calculations vs experiment)

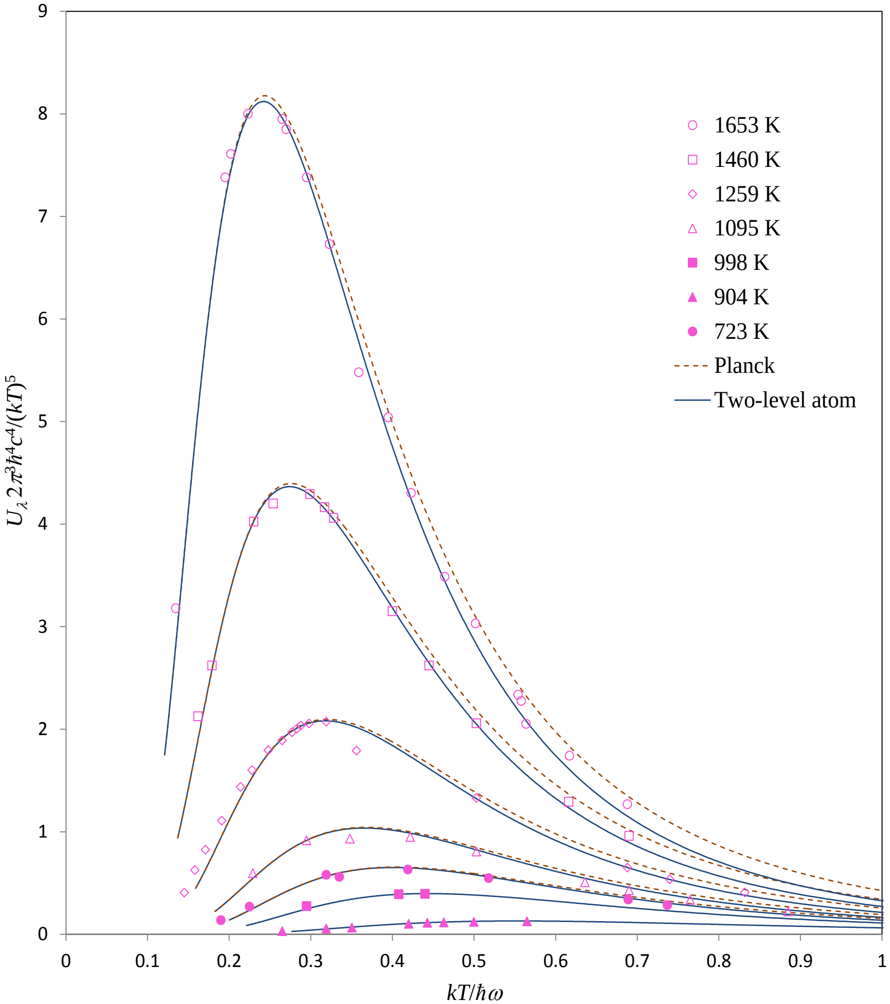


Figure. Comparison of the non-dimensional spectral energy density $U_\lambda$ calculated with the model of the two-level atom (solid line) and the experimental data (Lummer, O., & Pringsheim, 1901) (markers) for different temperatures. The dashed lines show the dependencies calculated by Planck's law

# Einstein A-coefficient

Optical equation for an atom in isotropic radiation field can be rewritten in the form

$$\frac{d\rho_{nn}}{dt} = -\frac{d\rho_{kk}}{dt} = B_{kn}U_\omega(\omega_{nk})\rho_{kk} - B_{kn}U_\omega(\omega_{nk})\rho_{nn} - A_{nk}\rho_{nn}\rho_{kk}$$

where

$$B_{kn} = B_{nk} = \frac{\pi^2 c^3}{\hbar\omega_{nk}^3}A_{nk}, \quad A_{nk} = 2\gamma_{nk} \qquad (1)$$

$\gamma_{nk} = \frac{2\omega_{nk}^3}{3\hbar c^3}|\mathbf{d}_{nk}|^2$ is the damping rate of spontaneous emission.

Let us introduce the notations

$$w_{kn}^{ind} = B_{kn}U_\omega(\omega_{nk}), \qquad w_{nk}^{ind} = B_{nk}U_\omega(\omega_{nk}), \qquad w_{nk}^{sp} = A_{nk}\rho_{kk}$$

Then

$$\frac{d\rho_{nn}}{dt} = -\frac{d\rho_{kk}}{dt} = w_{kn}^{ind}\rho_{kk} - w_{nk}^{ind}\rho_{nn} - w_{nk}^{sp}\rho_{nn}$$

This equation, formally, has the form of the kinetic equation describing the transitions of some fictitious system.

The “probabilities” of the corresponding “transitions” are connected by the relations

$$w_{kn}^{ind} = w_{nk}^{ind} = \frac{\pi^2 c^3}{\hbar\omega_{nk}^3}\frac{1}{\rho_{kk}}w_{nk}^{sp}U_\omega(\omega_{nk})$$

In the case when the excitation of the upper eigenmode $n$ is weak ($\rho_{nn} \ll \rho_{kk}$), one can take $\rho_{kk} \approx 1$, and then relation (1) becomes the well-known result of QED.

In this case the correct expression for the Einstein A-coefficient for spontaneous emission are obtained.

# Conclusion

The considered point of view allows explaining within the framework of classical field theory at least 15 basic phenomena underlying quantum mechanics which, it is believed, cannot be explained without quantization:
- thermal radiation;
- photoelectric effect;
- Compton effect;
- Born rule;
- Heisenberg uncertainty principle;
- stability of atoms;
- discrete emission spectra of atoms;
- spontaneous “transitions” and necessity of zero-point fluctuations;
- light-atom interactions;
- discrete events in the light-detector interaction;
- electrons diffraction and double-slit experiments with electrons;
- spin;
- Hanbury Brown and Twiss experiments;
- violation of the Bell inequalities;
- EPR and EPRB experiments;

At present, at least 3 basic phenomena remain still unexplained in the framework of classical field theory:
- Lamb shift;
- anomalous magnetic moment of electron;
- Stern and Gerlach experiment.

# Thank you for attention!

**You can find more details in:**


1. Rashkovskiy *S. A.* Are there photons in fact? *Proc. SPIE. 9570, The Nature of Light: What are Photons? VI*, 95700G (13 pages). (September 10, 2015) doi: 10.1117/12.2185010.
2. Rashkovskiy S.A. Quantum mechanics without quanta: the nature of the wave-particle duality of light. *Quantum Studies: Mathematics and Foundations*, (2016) 3:147–160, DOI: 10.1007/s40509-015-0063-5. See also arXiv:1507.02113 [quant-ph], 61p.
3. Rashkovskiy S.A. Semiclassical simulation of the double-slit experiments with single photons. *Progress of Theoretical and Experimental Physics*, 2015(12): 123A03 (16 pages) (2015), DOI: 10.1093/ptep/ptv162.
4. Rashkovskiy S.A. Quantum mechanics without quanta: 2. The nature of the electron. *Quantum Studies: Mathematics and Foundations* (2017), 4(1), 29-58. DOI: 10.1007/s40509-016-0085-7. See also arXiv:1507.02113 [quant-ph], 61p.
5. Rashkovskiy S.A. Classical-field model of the hydrogen atom. *Indian Journal of Physics*, (2017), 91(6), 607-621. DOI: 10.1007/s12648-017-0972-8. See also arXiv:1602.04090 [physics.gen-ph], 2016, 32 p.
6. Rashkovskiy S.A. Nonlinear Schrödinger equation and semiclassical description of the light-atom interaction. *Progress of Theoretical and Experimental Physics*, 2017(1): 013A03 (17 pages) (2017). DOI 10.1093/ptep/ptw177.
7. Rashkovskiy S.A. Classical-field theory of thermal radiation, arXiv:1604.05165 [physics.gen-ph], 2016, 32 p.