Presymplectic Currents and Weak Lagrangians for Massless Higher-Spin Fields

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Motivations

- At present the gauge HS theories are basically classical field theories defined by nonlinear (and non-Lagrangian) equations motion.
- The absence of a closed Lagrangian formulation constraints our understanding of quantum properties of HS fields.
- This gives impetus to a search for alternative quantization methods that are not so tightly related to the Lagrangin form of dynamics.

Covariant Phase Space and Presymplectic Structure

C. Crnković & E. Witten, 1987
G. J. Zuckerman, 1987

Plan:

1. Presymplectic geometry and mechanics
2. Covariant presymplectic structure in field theory
3. Symmetries, conservation laws and weak Lagrangians
4. Free HS fields in the unfolded representation
5. Covariant presymplectic structure for free HS fields

Presymplectic mechanics

A **presymplectic form** is a closed 2-form Ω on a smooth manifold M.

A vector field X (a function f) is called **Hamiltonian** if

$$i_X \Omega = df \,.$$

f is called the Hamiltonian of $X = X_f$.

If Ω is degenerate, the correspondence $f \leftrightarrow X_f$ is far from being 1-1:

$$X_f \quad \to \quad X_f + \ker \Omega \,.$$

The algebra of Hamiltonian functions is endowed with the Poisson bracket

$$\{f, g\} = i_{X_f} i_{X_g} \Omega \,.$$

The time evolution of a Hamiltonian function f is defined by the equation

$$\dot{f} = \{f, h\} \,,$$

where h is the Hamiltonian of the system.

Covariant phase space

In the covariant approach to the Hamiltonian mechanics the phase space of fields is identified with the solution space rather than the space of Cauchy data to the field equations.

$$S[\phi] = \int_V \mathcal{L}(\phi^i, \phi^i_\mu) d^n x\,, \qquad \phi^i_\mu = \partial_\mu \phi^i$$

$$\delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi^i_\mu} \right) \delta\phi^i \wedge d^n x + \int_{\partial V} \frac{\partial \mathcal{L}}{\partial \phi^i_\mu} \delta \phi^i \wedge d^{n-1} x_\mu$$

$$d^2 = \delta^2 = 0\,, \qquad d\delta = -\delta d\,, \qquad \partial_\mu \delta = \delta \partial_\mu$$

The boundary term gives the variational 1-form

$$\Theta[\phi, \delta\phi] = \int_\Sigma \frac{\partial \mathcal{L}}{\partial \phi^i_\mu} \delta\phi^i \wedge d^{n-1} x_\mu\,,$$

Σ being a Cauchy surface.

Covariant phase space

Let M denote the solution space, then the functional 2-form

$$\Omega = \delta\Theta = \int_\Sigma \left(\frac{\partial^2 \mathcal{L}}{\partial\phi^j \partial\phi^i_\mu} \delta\phi^j \wedge \delta\phi^i + \frac{\partial^2 \mathcal{L}}{\partial\phi^j_\nu \partial\phi^i_\mu} \delta\phi^j_\nu \wedge \delta\phi^i \right) \wedge d^{n-1}x_\mu$$

endows M with the presymplectic structure $\Omega|_M$.
Θ is called a **presymplectic potential**.

The presymplectic form Ω is on-shell independent of the choice of the Cauchy surface:

$$\Omega_\Sigma \approx \Omega_{\Sigma'}$$

The sign $\approx$ means equality on shell.

The presymplctic form Ω on M is degenerate iff the Lagrangian $\mathcal{L}$ is gauge invariant.

Generalization

Equations of motion for fields ϕ^i on an n-dimensional manifold N:

$$E_a(\phi, \partial\phi, \ldots, \partial^n\phi) = 0$$

Let Φ denote the space of all field configurations. Consider a "hybrid" differential form on $\Phi \times N$ of type $(2, n-1)$:

$$\mathit{\Omega} = \sum_{k,l=0}^{K,L} \mathit{\Omega}_{ij\mu_1\cdots\mu_{n-1}}^{\nu(k),\lambda(l)} \delta\phi^i_{\nu(k)} \wedge \delta\phi^j_{\lambda(l)} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n-1}} \,,$$

$\mathit{\Omega}$ defines a **presymplectic current** for the equations $E_a = 0$, if

$$\delta\mathit{\Omega} = 0 \,, \qquad d\mathit{\Omega} \approx 0 \,.$$

The presymplectic form is given by $\Omega = \int_\Sigma \mathit{\Omega}$.

[T.J. Bridges, P.E. Hydon, J.K. Lawson, Math. Proc. Camb. Phil. Soc., 2010]

Symmetries, charges and weak Lagrangians

Let $\delta\phi^i = V^i(\phi)$ be a global symmetry of equations of motion and the presymplectic structure, i.e.,

$$L_V E_a \approx 0\,, \qquad L_V \Omega \approx 0\,.$$

Then

$$i_V \Omega = \delta J\,, \qquad Q = \int_\Sigma J\,.$$

Q is a conserved charge and J is a conserved current, $dJ \approx 0$.

Since δ is acyclic,

$$\varOmega = \delta\varTheta \quad \Rightarrow \quad \delta d\varTheta \approx 0 \quad \Rightarrow \quad d\varTheta \approx \delta\mathcal{L}\,.$$

The n-form $\mathcal{L}$ is called **weak Lagrangian** for the equations of motion:

$$\frac{\delta\mathcal{L}}{\delta\phi^i} = U_i^a E_a\,.$$

[I. Khavkine, J. Math. Phys., 2013] [M. Grigoriev, arXiv:1606.07532]

Free HS fields in AdS_4

The Weyl algebra W:

$$f=\sum_{m,n}\frac{1}{m!n!}f_{\alpha(m)\dot\alpha(n)}(y^\alpha)^m(\bar y^{\dot\alpha})^n\,,\quad f_{\alpha(m)\dot\alpha(n)}\in\mathbb{C}\,,\quad \alpha,\dot\alpha=1,2.$$

$$f*g=\exp\left(i\epsilon^{\alpha\beta}\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial z^\beta}+i\epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar y^{\dot\alpha}}\frac{\partial}{\partial\bar z^{\dot\beta}}\right)f(y,\bar y)g(z,\bar z)|_{z=y}\,.$$

$$(f*g)*h=f*(g*h)\,,\qquad 1*f=f=f*1\,,\qquad \forall f,g,h\in W\,.$$

It is known that $so(3,2)\sim sp(4)$ and $sp(4)\subset Lie(W)$:

$$M_{\alpha\beta}=-\frac{i}{2}y_\alpha y_\beta\,,\qquad \bar M_{\dot\alpha\dot\beta}=-\frac{i}{2}\bar y_{\dot\alpha}\bar y_{\dot\beta}\,,\qquad P_{\alpha\dot\alpha}=-\frac{i}{2}y_\alpha\bar y_{\dot\alpha}$$

Free HS fields in AdS_4

Let Λ denote the exterior algebra of differential forms on AdS_4.

$$\Lambda\otimes W \ \ni\ F(y,\bar{y}|x,dx)=\sum_{m,n}\frac{1}{m!n!}F_{\alpha(m)\dot{\alpha}(n)}(x,dx)(y^{\alpha})^{m}(\bar{y}^{\dot{\alpha}})^{n}$$

The Lie superalgebra $Lie(W)$:

$$[F,G]_{\star}=F\star G-(-1)^{|F||G|}G\star F$$

The supertrace and invariant inner product:

$$\mathrm{Str}(F)=F(0,0|x,dx)\,,\qquad \langle F|G\rangle=\mathrm{Str}(F\star G)$$

$$\mathrm{Str}([F,G]_{\star})=0\,,\qquad \langle [H,F]_{\star}|G\rangle+(-1)^{|H||F|}\langle F|[H,G]_{\star}\rangle=0$$

$$\{F,G\}_{\star}=F\star G+(-1)^{|F||G|}G\star F$$

Vasiliev's equations for free HS fields

Weyl field: $C(y,\bar{y}|x) \in \Lambda^0 \otimes W$

Gauge field: $\omega(y,\bar{y}|x,dx) \in \Lambda^1 \otimes W$

The free HS field equations in unfolded representation:

$$D\omega + [h,\omega]_\star = \hat{H}_+ C_- + \hat{H}_- C_+\,, \qquad DC + \{h, C\}_\star = 0\,.$$

Here

$$D = d + [w, \cdot\,]\,, \quad w = \frac{1}{2} w^{\alpha\beta} M_{\alpha\beta} + \frac{1}{2}\bar{w}^{\dot\alpha\dot\beta}\bar{M}_{\dot\alpha\dot\beta}\,, \quad h = h^{\alpha\dot\alpha} P_{\alpha\dot\alpha}\,,$$

$$C_+ = C(y,0|x)\,, \qquad C_- = C(0,\bar{y}|x)\,,$$

$$\hat{H}_+ = h^{\gamma\dot\alpha} \wedge h_\gamma{}^{\dot\beta} \partial_{\dot\alpha}\partial_{\dot\beta}\,, \qquad \hat{H}_- = h^{\alpha\dot\gamma} \wedge h^\beta{}_{\dot\gamma} \partial_\alpha \partial_\beta\,.$$

h, w are the AdS vierbein and Lorentz connection.

HS presymplectic currents

Expansion in homogeneous polynomials in y and $\bar{y}$:

$$\omega = \sum_{m,n} \omega_{mn}\,, \qquad C = \sum_{m,n} C_{mn}\,,$$

The complex presymplectic currents

$$\Omega_{mn} = \begin{cases} \langle \delta\omega_{mn} | \hat{h}_+ \delta\omega_{m-1,n+1} \rangle, & \text{for } m > 0; \\ -\langle \delta\omega_{0,n} | \hat{H}_+ \delta C_{0,n+2} \rangle, & \text{for } m = 0. \end{cases}$$

$$\hat{h}_+ = h^{\alpha\dot{\alpha}} y_\alpha \partial_{\dot{\alpha}}\,, \qquad \hat{H}_+ = h^{\gamma\dot{\alpha}} \wedge h_\gamma{}^{\dot{\beta}} \partial_{\dot{\alpha}} \partial_{\dot{\beta}}\,.$$

$$\delta\Omega_{nm} = 0\,, \qquad d\Omega_{nm} \approx 0\,.$$

HS presymplectic currents

The most of the currents Ω_{mn} are on-shell equivalent to each other modulo d-exact currents:

$$\Omega_{mn} \simeq \Omega_{m-1,n+1} \qquad \forall m > 0 \,.$$

Furthermore,

$$\Omega_{mn} + \bar{\Omega}_{nm} \approx d\Psi_{mn} \,, \qquad \Psi_{mn} = -\frac{1}{2}\langle \delta\omega_{mn} | \delta\omega_{mn} \rangle \,.$$

Thus, there is essentially one presymplectic structure for each spin:

$$\Omega_s = \mathrm{Im}\, \Omega_{s-1,s-1} \,, \qquad s = 1, 2, 3, \dots$$

$$\Omega_s = \mathrm{Im}\, \Omega_{s-1/2,s-3/2} \,, \quad s = 3/2, 5/2, \dots$$

Low spins

Spin 1:

$$\Omega_1 = -\frac{1}{2i}\langle \delta\omega_{0,0}|\hat{H}_+\delta C_{0,2} - \hat{H}_-\delta C_{2,0}\rangle$$

Spin 2:

$$\Omega_2 = \frac{1}{2i}\langle \delta\omega_{1,1}|\hat{h}_+\delta\omega_{0,2} - \hat{h}_-\delta\omega_{2,0}\rangle$$

Spin 0 and 1/2:

$$\Omega_0 = \langle \delta C_{0,0}|\hat{H}_0\delta C_{1,1}\rangle\,, \qquad \Omega_{1/2} = \langle \delta C_{1,0}|\hat{\mathcal{H}}_+\delta C_{0,1}\rangle\,,$$

$$\hat{H}_0 = h^{\alpha\dot\beta}\wedge h_{\beta\dot\beta}\wedge h^{\beta\dot\alpha}\partial_\alpha\partial_{\dot\alpha}\,, \qquad \hat{\mathcal{H}}_+ = y_\alpha h^{\alpha\dot\beta}\wedge h_{\beta\dot\beta}\wedge h^{\beta\dot\alpha}\partial_{\dot\alpha}$$

Summary

- The unfolded representation of free HS fields admits a natural presymplectic structure.
- This presymplectic structure offers a far more flexible approach to the study and quantization of HS theories than the conventional Lagrangian formalism.
- In particular, it allows one to relate global symmetries with conservation laws and to define weak Lagrangians. The latter can be used for establishing the AdS/CFT correspondence.

Appendix A

Global AdS symmetry transformations:

$$\delta_\xi \omega = [\xi, \omega]_\star + 2\hat{H}^\xi_+ C_- + 2\hat{H}^\xi_- C_+ \,, \qquad \delta_\xi C = [\xi'', C]_\star + \{\xi', C\}_\star \,,$$

$$\xi = \xi' + \xi'' \,,$$

$$\xi' = -\frac{i}{2}\xi^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} \in \mathcal{F}^0_{1,1} \,, \qquad \xi'' = -\frac{i}{4}\xi^{\alpha\beta} y_\alpha y_\beta - \frac{i}{4}\xi^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \in \mathcal{F}^0_{2,0} \oplus \mathcal{F}^0_{0,2}$$

$$\hat{H}^\xi_- = \xi_\alpha{}^{\dot{\alpha}} h^{\alpha\dot{\beta}} \partial_{\dot{\alpha}} \partial_{\dot{\beta}} \,, \qquad \hat{H}^\xi_+ = \xi^\alpha{}_{\dot{\alpha}} h^{\beta\dot{\alpha}} \partial_\alpha \partial_\beta \,.$$

$$\mathcal{D}\xi = D\xi + [h, \xi] = 0 \,.$$

The conserved currents:

$$J_s = -\mathrm{Im}\langle \omega_{mn} | \hat{\xi}'' \hat{h}_+ \omega_{m-1,n+1} + \hat{h}_+ \hat{\xi}'_- \omega_{mn} \rangle$$
$$-\mathrm{Im}\langle \omega_{m+1,n+1} | \hat{\xi}'_+ \hat{h}_+ \omega_{m-1,n+1} \rangle$$

$m = n = s - 1$ or $m = s - 1/2$, $n = s - 3/2$.

Appendix B

$$L_{mn} = \frac{1}{2}\langle \hat{h}_+\omega_{m-1,n+1}|\hat{h}_+\omega_{m-1,n+1}\rangle + \frac{1}{2}\langle \hat{h}_-\omega_{mn}|\hat{h}_-\omega_{mn}\rangle$$

$$-\langle E^{\omega}_{m-1,n+1}|\hat{h}_-\omega_{mn}\rangle\,, \qquad m>0\,,$$

$$L_{0,n} = \langle \omega_{0,n}|\hat{H}_+E^C_{0,n+2}\rangle - \frac{1}{2}\langle \hat{H}_+C_{0,n+2}|\hat{H}_+C_{0,n+2}\rangle\,.$$

The weak Lagrangian:

$$\mathcal{L}_{mn} = \operatorname{Im} L_{mn}\,,$$

$$S = \int_{AdS_4}\mathcal{L}\,, \quad \mathcal{L} = \sum_{n,m=0}^{\infty}\lambda_{nm}\mathcal{L}_{nm}\,, \qquad \lambda_{nm}\in\mathbb{R}\,.$$

$$\mathcal{L}_{nm} \simeq \Lambda_s = -\frac{1}{2}\operatorname{Im}\langle \hat{H}_+C_{0,2s}|\hat{H}_+C_{0,2s}\rangle\,, \qquad n+m=2s-2\,.$$