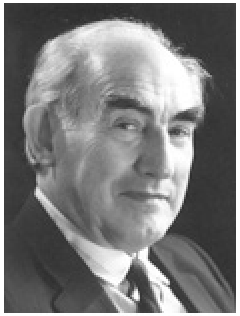

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Passive scalar transport by a non-Gaussian turbulent flow (Batchelor regime)

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Passive scalar transport

(toy model for turbulence, anomalous scaling)

$$\frac{\partial\theta}{\partial t} + \boldsymbol{V} \cdot \nabla\theta = \varkappa\, \Delta\theta$$

$\boldsymbol{V}$ is a random velocity field

Kraichnan model: $\boldsymbol{V}$ δ –correlated in time, Gaussian and scale-invariant in space

- well studied, many results. But far from reality (no energy cascade)

$$B = \int A dt \xrightarrow{?} B_G$$

$$\langle B_G \rangle = \langle B \rangle = \langle A \rangle t$$

$$D(B_G) = D(B) = D(A)\,\mathrm{t}$$

$$\langle B^n \rangle \approx \langle B_G{}^n \rangle \quad \text{but} \quad \langle e^{nB} \rangle \neq \langle e^{nB_G} \rangle \quad !$$

Non-Gaussian velocity field

Batchelor limit: $A_{ij} = \dfrac{\partial V_i}{\partial r_j}$ (viscous range, $\varkappa \ll \nu$)

A_{ij} - stationary random process; Lagrangian frame

Balkovsky, Fouxon 1999: $\langle \theta^{\alpha} \rangle = e^{\gamma_\alpha t}$

Saturation of γ_α at large α.

No relation between θ and A_{ij} statistics.

Il'yn, Sirota, Zybin 2017 (submitted to PRE):

Exact expressions for γ_α in terms of A_{ij} statistics.

Universal saturation: γ_α =const if $\alpha \geq 2$

(quasi)-Lagrangian frame: $$\frac{\partial\theta}{\partial t}+\frac{\partial\theta}{\partial r_i}A_{ij}r_j=\varkappa\,\Delta\theta \qquad tr\;A=0$$

$A_{ij}(t)$ -stationary random process with known statistics

(for δ -correlated velocity, statistics of $A_{ij}(t)$ is the same in Euler and Lagrangian frames)

$$\begin{array}{ccc} \theta(\boldsymbol{r},t) & \xrightarrow{\text{Fourier}} & \theta(\boldsymbol{k},t) \\ \boldsymbol{r}'=\boldsymbol{Q}\boldsymbol{r}\ \downarrow & & \downarrow\ \boldsymbol{k}=\boldsymbol{p}\boldsymbol{Q} \\ \theta(\boldsymbol{r}',t) & \xrightarrow{\text{Fourier}} & \theta(\boldsymbol{p},t) \end{array}$$

$$\dot{\boldsymbol{Q}}=-\boldsymbol{Q}\boldsymbol{A}\ ,\qquad \boldsymbol{Q}(0)=\boldsymbol{1}$$

$$\frac{\partial}{\partial t}\theta(\boldsymbol{p},t)=-\varkappa\;p_m\left(QQ^T\right)_{mn}p_n\;\theta$$

$$\theta(\boldsymbol{p},t)=\theta(\boldsymbol{p},0)\;e^{-\varkappa\;p_m\int\left(QQ^T\right)_{mn}(t')dt'p_n}$$

$$\theta(t)\equiv\theta(\boldsymbol{r}=0,t)=\int\theta(\boldsymbol{p},t)\;d^3p$$

$$\theta(\boldsymbol{p},t)=\theta(\boldsymbol{p},0)\; e^{-\varkappa\; p_m \int \left(QQ^T\right)_{mn}(t')dt' p_n}$$

Isolated blob: $\theta(\boldsymbol{p},0)=e^{-l^2\boldsymbol{p}^2}$

$$\theta(t)=\int \theta(\boldsymbol{p},t)\; d^3p=\int e^{-p_m D_{mn} p_n} d^3p=(detD)^{-1/2}$$

where $D_{mn}=\varkappa\int \left(QQ^T\right)_{mn}(t')dt'+l^2\delta_{mn}$

$$\langle \theta^\alpha \rangle = \langle (detD)^{-\alpha/2} \rangle$$

Homogenous initial condition:

$$\left\langle \theta(\boldsymbol{p},0)\theta(\boldsymbol{p}',0)\right\rangle = e^{-l^2\boldsymbol{p}^2}\delta(\boldsymbol{p}-\boldsymbol{p}')$$

$$\langle \theta^\alpha \rangle = \langle (detD)^{-\alpha/4} \rangle$$

Evolution matrix: Iwasawa decomposition

$$\boldsymbol{Q} = \boldsymbol{Z}\ \boldsymbol{d}\ \boldsymbol{R}$$

$$Z_{ij} = 0 \quad if \;\; i > j, \qquad Z_{jj} = 1$$

$$\boldsymbol{d} = diag(d_1, \ldots, d_N)\ ,\quad d_j > 0\ ,\quad d_1 \ldots d_N = 1$$

$$R \in SO(3)$$

$$\dot{\boldsymbol{Q}} = -\boldsymbol{Q}\boldsymbol{A}\ ,\quad \boldsymbol{Q}(0) = \boldsymbol{1}$$

Furstenberg 1963, Tutubalin 1978,...

$t \to \infty$

$$\exists \quad \boldsymbol{Z}_\infty = \lim_{t\to\infty} \boldsymbol{Z}$$

$\boldsymbol{Z}_\infty$ depends on the realization

$$\exists \quad \lambda_i = \lim_{t\to\infty} \frac{\ln d_i(t)}{t}\ , \qquad \lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$$

λ_i are universal for given process

$$D_{mn} = \varkappa \int \left(QQ^T\right)_{mn}(t')dt' + l^2\delta_{mn} = \varkappa \int \left(\boldsymbol{Z}\boldsymbol{d}^2\boldsymbol{Z}^T\right)(t')dt' + l^2\boldsymbol{I}$$

(R vanishes!)

$$t \to \infty\ : \qquad \det D = \left(\varkappa\, \boldsymbol{d}^2 + \boldsymbol{C}(l^2, \varkappa)\right) \propto d_1^{\,2\mu_1} \ldots d_N^{\,2\mu_N}$$

$$\mu_j = \begin{matrix} 1 & if\ \ \mathrm{d}_j \gg 1 \\ 0 & if\ \ \mathrm{d}_j \le 1 \end{matrix}$$

$$\langle \theta^\alpha \rangle = \langle (detD)^{-\alpha/2} \rangle \propto \left\langle d_1^{\,-\alpha\mu_1} \ldots d_N^{\,-\alpha\mu_N} \right\rangle$$

Cumulant functions and functionals

Random variable z $\longrightarrow$ $$e^{W_Z(\boldsymbol{k})} = \left\langle e^{i\,\boldsymbol{k}\,\boldsymbol{z}} \right\rangle = \int P(\boldsymbol{z}) e^{i\boldsymbol{k}\boldsymbol{z}} d\boldsymbol{z}$$

Random process $\rho(t)$ $\longrightarrow$ $$e^{W_\rho[\boldsymbol{\eta}(t)]} = \left\langle e^{i\int \mathrm{tr}\,\boldsymbol{\eta}(t)\boldsymbol{\rho}(t)dt} \right\rangle$$

$$W[\boldsymbol{\eta}(t)] = \int w\big(\boldsymbol{\eta}(t)\big)dt$$ for slow $\eta(t)$

$$\frac{\ln d_i}{T} = z_i = \frac{1}{T}\int_0^T \rho_i dt$$ $\Longrightarrow$ $$W_z(k) = w_\rho(k/T)\cdot T$$

$$z_i \xrightarrow[t\to\infty]{} \langle \rho_i \rangle = \lambda_i$$

$$\langle \theta^\alpha \rangle \propto \left\langle d_1^{-\alpha\mu_1} \ldots d_N^{-\alpha\mu_N} \right\rangle \propto \left\langle e^{-\alpha(\mu_1 z_1 + \cdots + \mu_N z_N)T} \right\rangle$$

$$\mu_j(z) = \begin{matrix} 1 & if\ z_j > 0 \\ 0 & if\ z_j \le 0 \end{matrix}$$

$$\langle \theta^\alpha \rangle \propto \left\langle e^{-\alpha(\mu_1 z_1 + \cdots + \mu_N z_N)T} \right\rangle \qquad \mu_j(z) = \begin{matrix} 1 & if\ z_j > 0 \\ 0 & if\ z_j \le 0 \end{matrix} \qquad z_i = \frac{1}{T}\int_0^T \rho_i dt \xrightarrow[T\to\infty]{} \langle \rho_i \rangle = \lambda_i$$

$$\lambda_1 \le \lambda_2 \le 0 \le \lambda_K \ldots \le \lambda_N$$

Most naïve consideration: $z_i \approx \lambda_i$

$$\langle \theta^\alpha \rangle \sim e^{-\alpha(\lambda_K + \cdots + \lambda_N)T}$$

(valid only for very small α)

Less naïve consideration: $\mathrm{sign}\ z_i = sign\ \lambda_i$

$$e^{W_\rho[\boldsymbol{\eta}(t)]} = \left\langle e^{i\int \mathrm{tr}\,\boldsymbol{\eta}(t)\boldsymbol{\rho}(t)dt} \right\rangle$$

$$\langle \theta^\alpha \rangle \sim \left\langle e^{-\alpha(z_K + \cdots + z_N)T} \right\rangle = e^{w_\rho(0,\ldots,0,i\alpha,\ldots,i\alpha)T}$$

(valid only for α small enough)

Right consideration:

$$w_\rho(k) \xrightarrow{\text{saddle point}} P(z) \xrightarrow{\text{saddle point}} \langle \theta^\alpha \rangle \propto e^{\gamma_\alpha T} \,,$$

$$\gamma_\alpha = \max_{k^*} \phi(k^*) \,, \qquad \phi = -ik_j^* z_j + w_\rho(k^*) - \alpha \sum_{j=1}^{N} \mu_j(z)\, z_j \,, \qquad z_j = -i \left.\frac{\partial w_\rho}{\partial k_j}\right|_{k^*}$$

$z_i \to \lambda_i$ but for some realizations $z_i(T) \le 0$ even if $\lambda_i > 0$.

The bigger α, the more is their contribution to $\langle \theta^\alpha \rangle$.

$$\langle \theta^{\alpha} \rangle \propto e^{\gamma_\alpha T} \qquad \mu_j(z) = \begin{cases} 1 & if \ z_j > 0 \\ 0 & if \ z_j \le 0 \end{cases}$$

$$\gamma_\alpha = \max_{k^*} \phi(k^*) \ , \qquad \phi = -ik_j^* z_j + w_\rho(k^*) - \alpha \sum_{j=1}^{N} \mu_j(z) \ z_j \ , \qquad z_j = -i \left. \frac{\partial w_\rho}{\partial k_j} \right|_{k^*}$$

Inside the regions of constant μ_j :

$$\frac{\partial \phi}{\partial k^*_j} = 0 \longrightarrow k_j^* = i\alpha\mu_j$$

(='less naïve consideration')

On the boundary $z_n = 0$:

$$\left. \frac{\partial w_\rho}{\partial k_n} \right|_{k^*} = 0 \ , \qquad j \neq n : \ k_j^* = i\alpha\mu_j$$

In both cases,

$$\langle \theta^{\alpha} \rangle = e^{w_\rho(k^*)T}$$

Relation between $\boldsymbol{A}$ and $\boldsymbol{\rho}$

Il'yn, Sirota, Zybin 2016:

$$\dot{\boldsymbol{Q}} = -\boldsymbol{Q}\boldsymbol{A} \quad \Longrightarrow \quad -\boldsymbol{A} = \boldsymbol{Q}^{-1}\dot{\boldsymbol{Q}} = \boldsymbol{R}^{T}\boldsymbol{d}^{-1}\boldsymbol{Z}^{-1}\frac{d}{dt}(\boldsymbol{Z}\boldsymbol{d}\boldsymbol{R})$$

Convenient variable: $\boldsymbol{X} = \boldsymbol{R}\boldsymbol{Q}^{-1}\dot{\boldsymbol{Q}}\boldsymbol{R}^{T} = \underbrace{\boldsymbol{d}^{-1}\dot{\boldsymbol{d}}}_{\text{diagonal}} + \boldsymbol{d}^{-1}\boldsymbol{Z}^{-1}\dot{\boldsymbol{Z}}\boldsymbol{d} + \dot{\boldsymbol{R}}\boldsymbol{R}^{T}$

Then $\boldsymbol{X} = -\boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^{T}$ $\qquad X_{nn} = -d_n^{-1}\dot{d}_n = \rho_n$

$$w_X(k) = w_A(ik_0 - k) - w_A(ik_0)$$

$$(k_0)_{ij} = \frac{2j-1-N}{2}\delta_{ij}$$

$$w_\rho(k_1,\ldots,k_N) = w_X(k)\Big|_{k_{ij} = \begin{cases} k_i, & i = j \\ 0, & i \neq j \end{cases}}$$

Three-dimensional case

$$\xi = i(k_1^* - k_2^*)\,, \quad \eta = i(k_1^* - k_3^*)\,, \quad \psi = i(k_1^* + k_2^* + k_3^*)$$

Incompressibility $\longrightarrow$ $\partial w_\rho / \partial \psi = 0\,, \quad w_\rho = w_\rho(\xi, \eta)$

$$z_1 = \frac{\partial w_\rho}{\partial \xi} + \frac{\partial w_\rho}{\partial \eta}, \quad z_2 = -\frac{\partial w_\rho}{\partial \xi}, \quad z_3 = -\frac{\partial w_\rho}{\partial \eta}$$

$$z_j = -i \frac{\partial w_\rho}{\partial k_j^*}$$

Inside the regions of constant μ_j :

$$k_j^* = i\alpha\mu_j\,, \quad \xi^* = \alpha(\mu_2 - \mu_1)\,, \quad \eta^* = \alpha(\mu_3 - \mu_1)$$

$$\mu_j(z) = \begin{matrix} 1 & if \;\; z_j > 0 \\ 0 & if \;\; z_j \le 0 \end{matrix}$$

On the boundary $z_2 = 0$: $\dfrac{\partial w_\rho}{\partial \xi^*} = 0 \;\;, \eta^* = \alpha(\mu_3 - \mu_1)$

$$w_\rho(\xi, \eta) = w_A(1 - \xi, 2 - \eta) - w_A(1, 2)$$

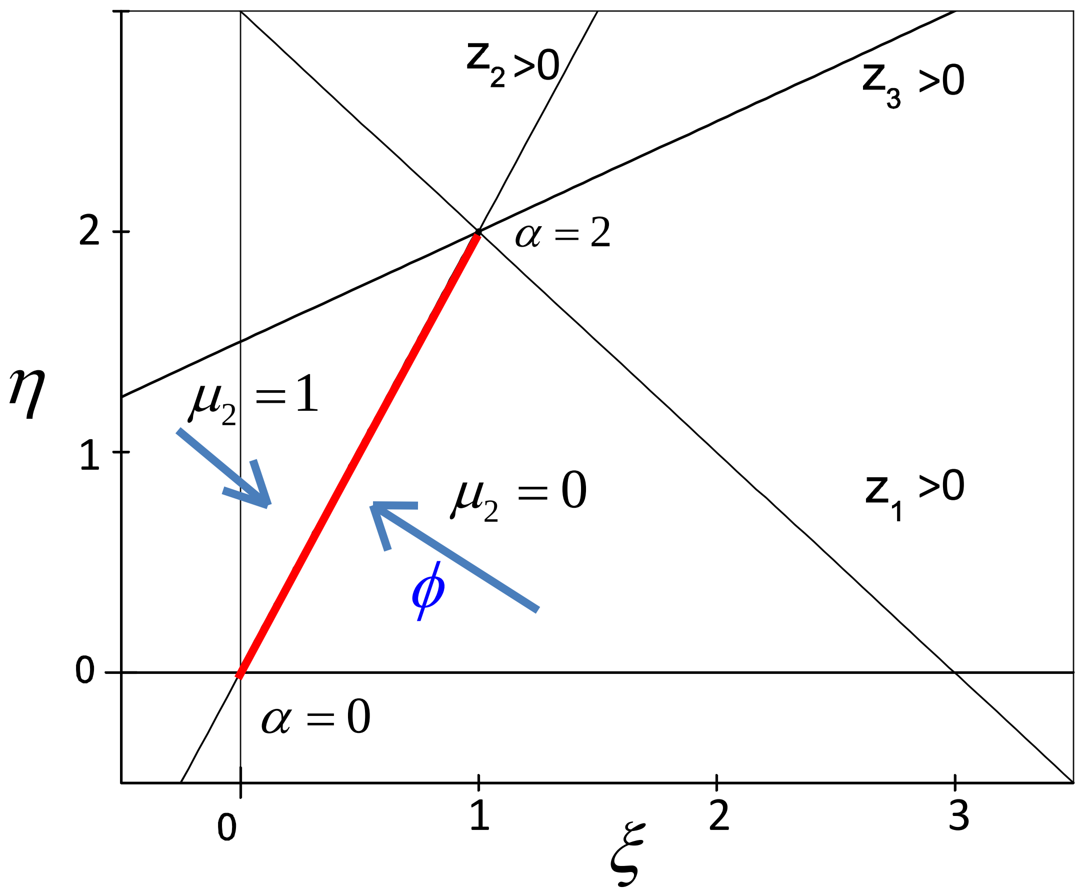
Examples: Gaussian statistics

$$w_A^G = D\left(\frac{1}{3}(k_1+k_2+k_3)^2-(k_1^2+k_2^2+k_3^2)\right)=\frac{2}{3}D(\xi^2+\eta^2-\xi\eta)$$

$$w_\rho(\xi,\eta)=w_A(1-\xi,2-\eta)-w_A(1,2)$$

$$w_A(0,0)=w_\rho(0,0)=0$$

$$\min w_\rho:\ \xi=1,\ \eta=2$$



$$\gamma_\alpha=\max_{\xi,\eta}\phi(\xi,\eta)$$

Vicinity of the boundary:

$$\mu_1=0\ ,\quad \mu_3=1$$

$$\xi^*=\alpha\mu_2,$$

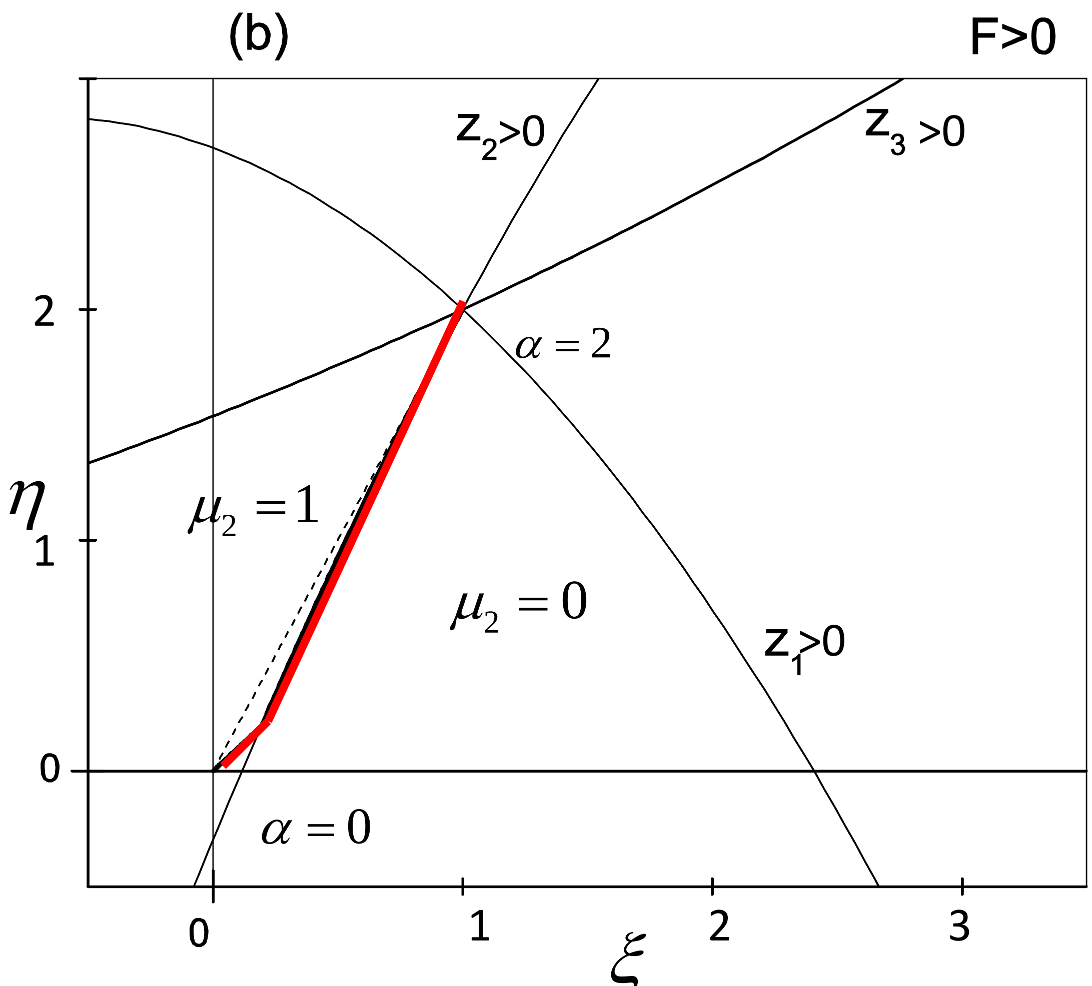
$$\eta^*=\alpha$$

$\alpha=2$ *-saturation*

Examples: Small deviation from Gauss

$$w_A = \frac{2}{3} D(\xi^2 + \eta^2 - \xi\eta) - \frac{2}{9} F(\xi^3 + \eta^3) + \frac{1}{3} F\xi\eta(\xi + \eta) , \quad F > 0$$

$$w_\rho(\xi,\eta) = w_A(1-\xi, 2-\eta) - w_A(1,2)$$



$$w_A(0,0) = w_\rho(0,0) = 0$$

$$\min w_\rho : \ \xi = 1, \ \eta = 2$$

Vicinity of the boundary:

$$\mu_1 = 0 , \quad \mu_3 = 1 \qquad \xi^* = \alpha\mu_2 ,$$

$$\eta^* = \alpha$$

$$\gamma_\alpha = -2D + \frac{1}{2} D(\alpha - 2)^2$$

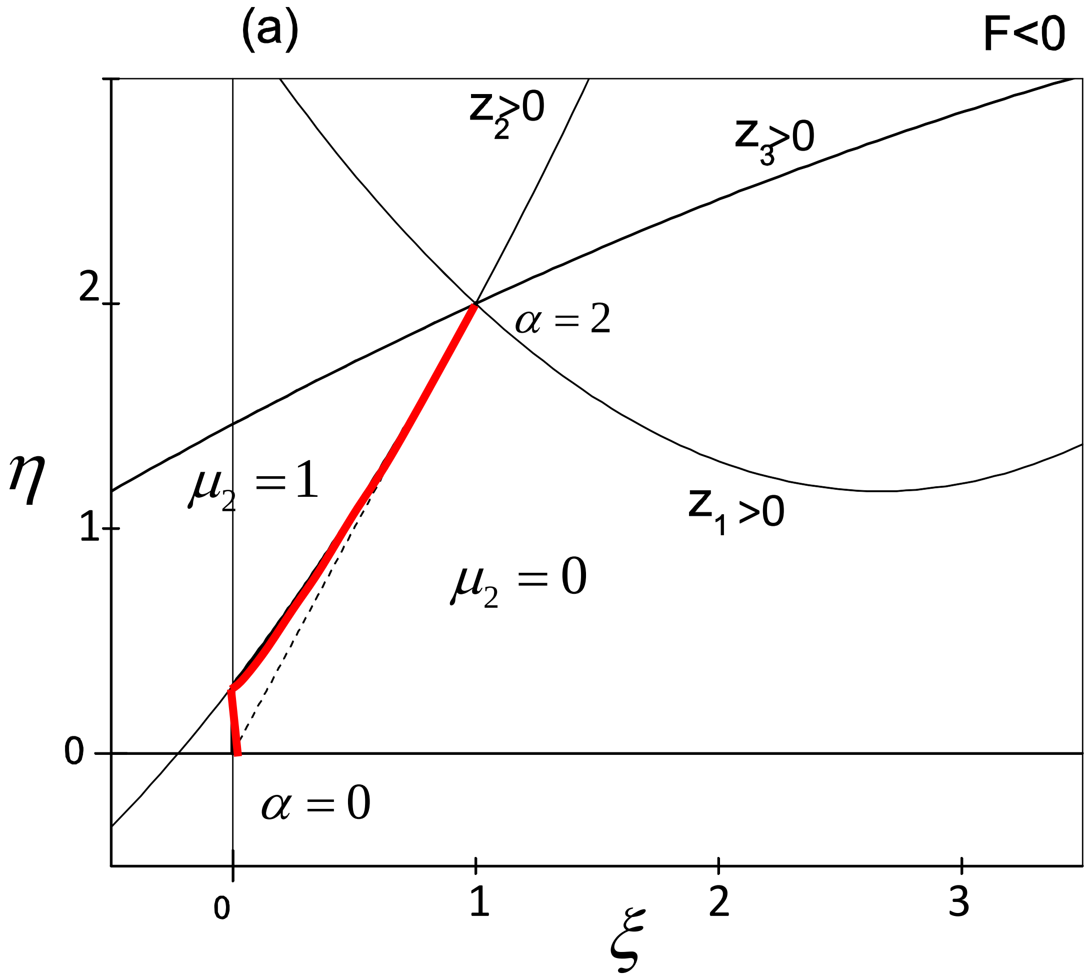
$$-\frac{3}{32} D\left(\frac{F}{D}\right)^2 (\alpha - 2)^4 + O\left((F/D)^4\right)$$

$\alpha = 2$ *-saturation is universal!*

Examples: Small deviation from Gauss

$$w_A = \frac{2}{3}D(\xi^2+\eta^2-\xi\eta) - \frac{2}{9}F(\xi^3+\eta^3) + \frac{1}{3}F\xi\eta(\xi+\eta)\,, \quad F<0$$

$$w_\rho(\xi,\eta) = w_A(1-\xi, 2-\eta) - w_A(1,2)$$



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Vicinity of the boundary:

$$\mu_1 = 0\,, \quad \mu_3 = 1 \qquad \xi^* = \alpha\mu_2, \quad \eta^* = \alpha$$

$$\gamma_\alpha = -2D + \frac{1}{2}D(\alpha-2)^2 - \frac{3}{32}D\left(\frac{F}{D}\right)^2(\alpha-2)^4 + O\left((F/D)^4\right)$$

$\alpha = 2$ -saturation is universal!

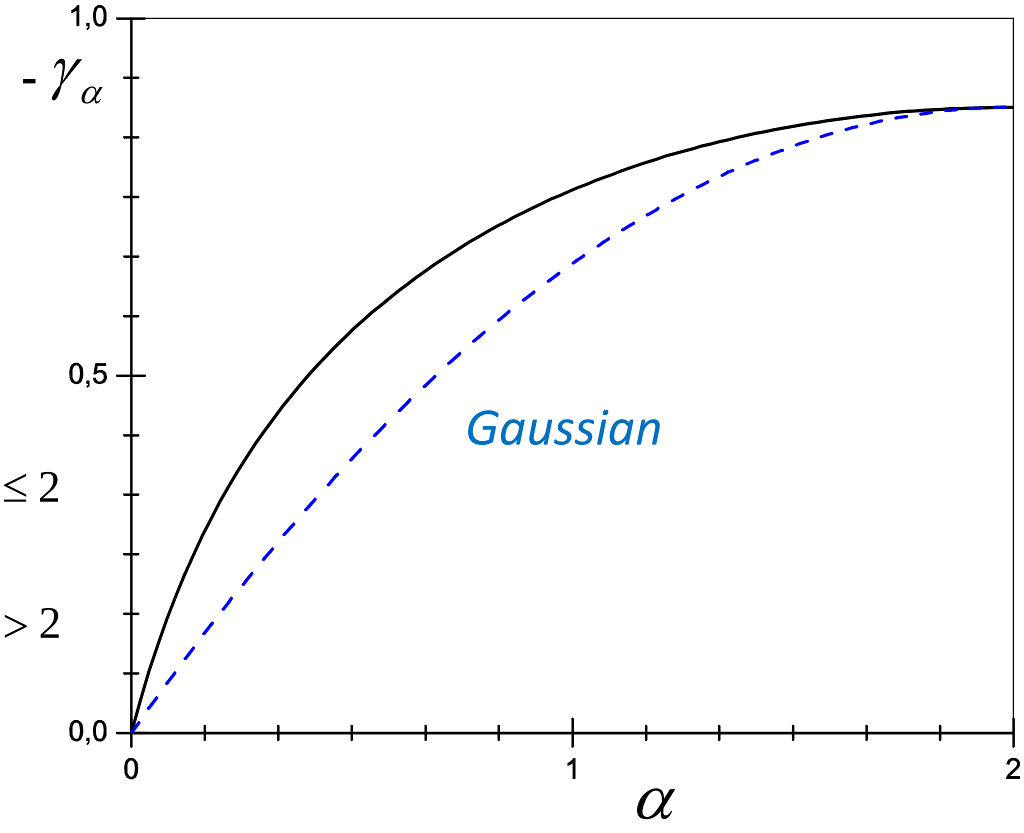
Examples: Exponential statistics

$$f(A_1, A_2, A_3) = f_0 e^{-c\sqrt{A_1^2 + A_2^2 + A_3^2}} \delta(A_1 + A_2 + A_3)$$

$$w_A = -\frac{1}{2}\ln\left(3c^2 - 4\left(\xi^2 + \eta^2 - \xi\eta\right)\right) + \frac{1}{2}\ln\left(3c^2\right)$$

$$w_\rho(\xi,\eta) = w_A(1-\xi, 2-\eta) - w_A(1,2)$$

$$\gamma_\alpha = \begin{cases} \phi(\alpha/2, \alpha) = \frac{1}{2}\ln\frac{c^2-4}{c^2-(\alpha-2)^2}, & \alpha \le 2 \\ \phi(1,2) = \frac{1}{2}\ln\frac{c^2-4}{c^2}, & \alpha > 2 \end{cases}$$



Summary

- We analyze the passive scalar advection in a turbulent flow at time much bigger than the correlation time of the flow, and scales much smaller than the viscous scale (viscous range); diffusivity is assumed to be much smaller than viscosity, linear approximation for the velocity field is valid (Batchelor regime).
- We obtain exact expressions for the exponents γ_α of the Lagrangian scalar density moments in terms of Lagrangian velocity strain tensor statistics. (If velocity is assumed to be δ -correlated, the latter coincides with the Eulerian strain tensor statistics).
- The exponents saturate at the universal value α*=2*, independently of the statistics. In the range between α*=2* and α*=2* they can differ significantly from those in *the Gaussian case.*