

# Spinning Witten Diagrams

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based on work with M. Taronna  
1603.0022, 1702.08619 [hep-th]

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# Introduction

[Gubser et al., Witten]

CFT<sub>d</sub> correlation  
functions



AdS<sub>d+1</sub> scattering  
processes

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle =$$

tree level

- For external scalars: Well understood how to evaluate

[<sup>98</sup> Mück et al., Freedman et al., Liu et al., Arutyunov et al., Petkou et al., ...]

- More recently, the underlying structure: CPWE of Witten diagrams

[Costa et al.; Bekaert, Erdmenger, Ponomarev, C.S.; Hijano et al.]

# Introduction

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CFT<sub>d</sub> correlation  
functions



AdS<sub>d+1</sub> scattering  
processes

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \text{[diagram 1]} + \text{[diagram 2]} + \text{[diagram 3]} + \text{[diagram 4]} + \text{[diagram 5]} + \dots$$

The diagram shows a series of Feynman diagrams on a sphere. The first diagram is a simple rectangle with four external legs. The second and third diagrams are enclosed in a green box labeled "tree level" and represent more complex tree-level diagrams with internal lines. The fourth and fifth diagrams are loop-level diagrams. The sequence continues with an ellipsis.

Today: With spinning external (& internal) legs!

- For external scalars: Well understood how to evaluate

[98 Mück et al., Freedman et al., Liu et al., Arutyunov et al., Petkou et al., ...]

- More recently, the underlying structure: CPWE of Witten diagrams

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# Introduction

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The diagram shows a series of Feynman diagrams on a sphere. The first diagram is a tree-level diagram with four external legs. The second and third diagrams are also tree-level diagrams with four external legs, enclosed in a green box labeled "tree level". The fourth and fifth diagrams are loop-level diagrams with four external legs. The ellipsis indicates that there are more diagrams in the series.

Today: Spinning external (& internal) legs!

## Motivations:

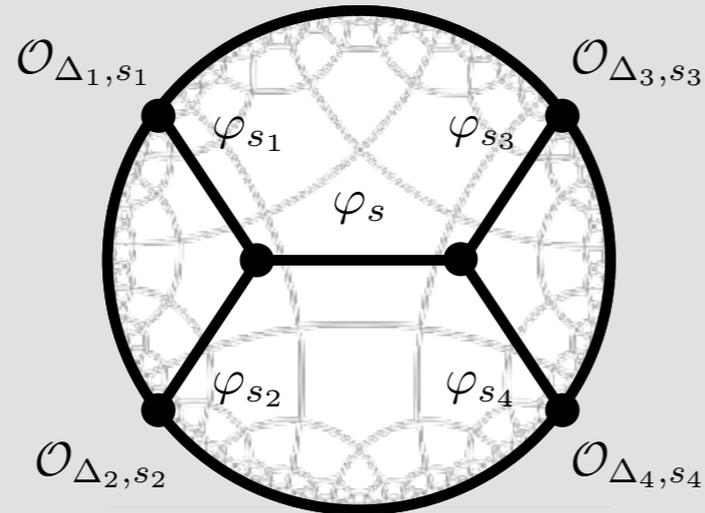
- Holographic reconstruction
- Conformal Bootstrap

# Outline

- Conformal Partial Wave Expansion (CPWE) of Spinning Witten Diagrams
- Spinning 3pt
- Spinning 4pt exchange diagrams

# Goal

Decompose

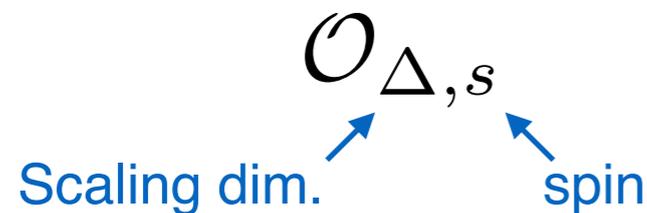


into Conformal Partial Waves

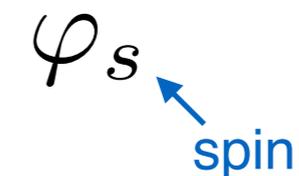
Totally symmetric external fields  $\varphi_{s_i}$  and internal field  $\varphi_s$

**Field - Operator map:**

**single-trace operators**



**bulk fields (single-particles)**



$$\text{mass: } m_s^2 R^2 = \Delta (\Delta - d) - s$$

# General Approach

CFT side.

# General Approach

## Review: CPWE of 4pt CFT correlators

$\Delta$  and  $s$  = scaling dim. and spin of  $\mathcal{O}$

$$\langle \mathcal{O}_1(y_1) \mathcal{O}_2(y_2) \mathcal{O}_3(y_3) \mathcal{O}_4(y_4) \rangle = \sum_{\{\Delta, s\}} c_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2} c_{\mathcal{O}\mathcal{O}_3\mathcal{O}_4} W_{\Delta, s}(y_1, y_2; y_3, y_4)$$

- Operator Product Expansion (OPE) coefficients

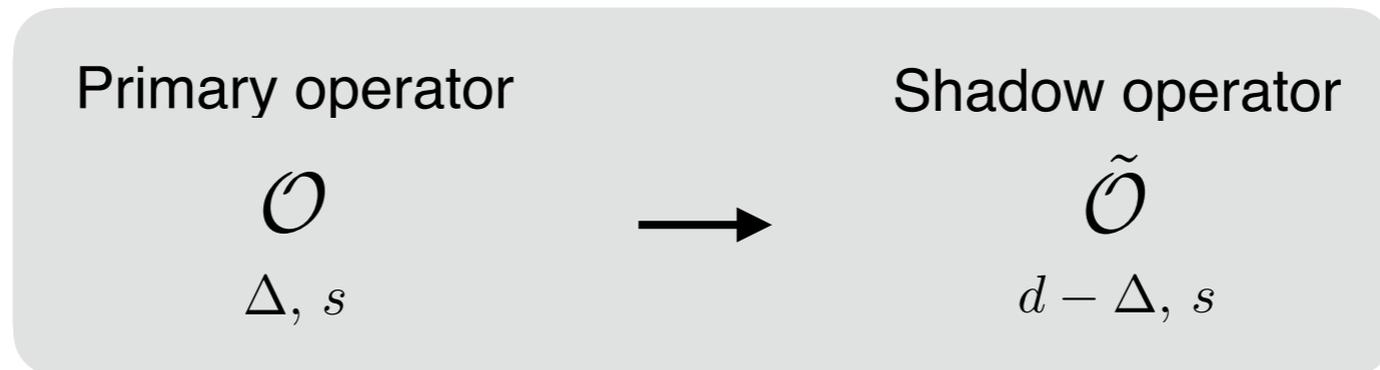
$$\mathcal{O}_1 \times \mathcal{O}_2 \sim c_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2} \mathcal{O} + \dots \quad \text{and} \quad \mathcal{O}_3 \times \mathcal{O}_4 \sim c_{\mathcal{O}\mathcal{O}_3\mathcal{O}_4} \mathcal{O} + \dots$$

- Conformal Partial Waves (CPWs)  $W_{\Delta, s}$  :
  - Re-sums contributions from  $\mathcal{O}$  and all descendants  $\partial \dots \partial \mathcal{O}$
  - **For external scalars:** Explicit expressions in terms of elementary functions only possible in even  $d$
  - **For spinning external operators:** Comparably little known

# General Approach

## Shadow Formalism [Ferrara et al.]

$$\tilde{\mathcal{O}}(y) = \kappa \int d^d y' \frac{I(y-y')}{[(y-y')^2]^{d-\Delta}} \mathcal{O}(y')$$



Projector:  $\mathcal{P}_{\Delta, s} = \kappa \int d^d y \mathcal{O}(y) |0\rangle \langle 0| \tilde{\mathcal{O}}(y)$  {

- dimensionless
- conformally invariant

Projects onto multiplet of  $\mathcal{O}$  and  $\tilde{\mathcal{O}}$ :

$\rightarrow \langle \mathcal{O}_1(y_1) \mathcal{O}_2(y_2) \mathcal{P}_{\Delta, s} \mathcal{O}_3(y_3) \mathcal{O}_4(y_4) \rangle = c_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2} c_{\mathcal{O}\mathcal{O}_3\mathcal{O}_4} W_{\Delta, s}(y_1, y_2; y_3, y_4) + \text{shadow}$

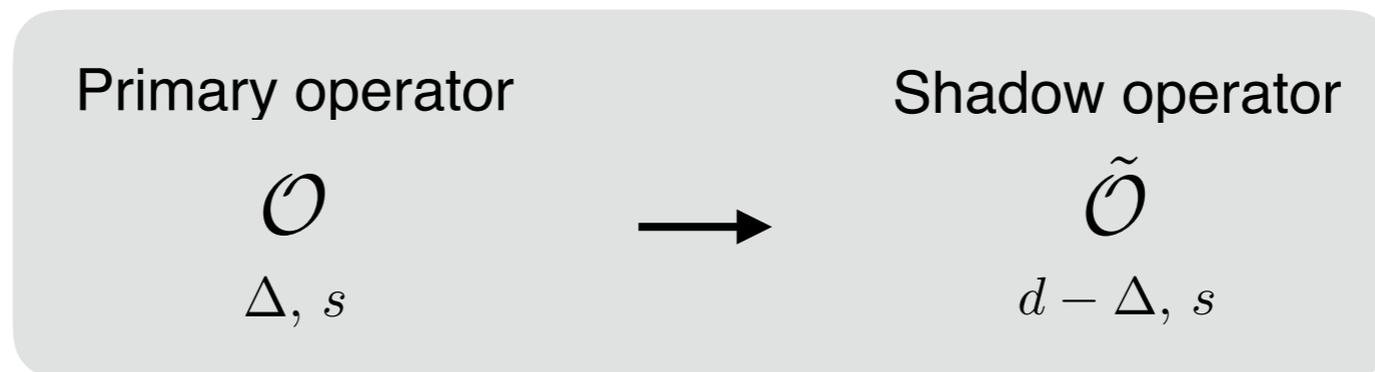


$\kappa \int d^d y \langle \mathcal{O}_1(y_1) \mathcal{O}_2(y_2) \mathcal{O}(y) \rangle \langle \tilde{\mathcal{O}}(y) \mathcal{O}_3(y_3) \mathcal{O}_4(y_4) \rangle$  **factorisation into 3pt correlators!**

# General Approach

## Shadow Formalism [Ferrara et al.]

$$\tilde{\mathcal{O}}(y) = \kappa \int d^d y' \frac{I(y-y')}{[(y-y')^2]^{d-\Delta}} \mathcal{O}(y')$$



3pt conformal structure



Leads to integral expression for CPWs:

[notation:  $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O} \rangle = c_{\mathcal{O} \mathcal{O}_1 \mathcal{O}_2} \langle \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O} \rangle \rangle$ ]

$$W_{\Delta, s} + W_{d-\Delta, s} = \int d^d y \langle \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}(y) \rangle \rangle \langle \langle \tilde{\mathcal{O}}(y) \mathcal{O}_3 \mathcal{O}_4 \rangle \rangle$$

### Useful because:

- **Universal:** Holds for any  $d$ , any  $\mathcal{O}$  and any external primary operators  $\mathcal{O}_i$
- In terms of 3pt conformal structures
- Naturally arise from **harmonic function decomposition of Witten diagrams**

# General Approach

## Shadow Formalism for Spinning CPWs

$$\tilde{\mathcal{O}}(y) = \kappa \int d^d y' \frac{I(y-y')}{[(y-y')^2]^{d-\Delta}} \mathcal{O}(y')$$

$$W_{\Delta,s}^{\mathbf{m},\mathbf{n}} + W_{d-\Delta,s}^{\mathbf{m},\mathbf{n}} = \kappa \int d^d y \langle\langle \mathcal{O}_{\Delta_1,s_1} \mathcal{O}_{\Delta_2,s_2} \mathcal{O}(y) \rangle\rangle^{(\mathbf{m})} \langle\langle \tilde{\mathcal{O}}(y) \mathcal{O}_{\Delta_3,s_3} \mathcal{O}_{\Delta_4,s_4} \rangle\rangle^{(\mathbf{n})}$$

[notation:  $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle = c_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2}\langle\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle\rangle$  ]

Basis of 3pt spinning conformal structures:

$$\tau_i = \Delta_i - s_i$$

$$\langle\langle \mathcal{O}_{\Delta_1,s_1}(y_1) \mathcal{O}_{\Delta_2,s_2}(y_2) \mathcal{O}_{\Delta_3,s_3}(y_3) \rangle\rangle^{(\mathbf{n})} = \frac{Y_1^{s_1-n_2-n_3} Y_2^{s_2-n_1-n_3} Y_3^{s_3-n_1-n_2} H_1^{n_1} H_2^{n_2} H_3^{n_3}}{(y_{12}^2)^{\frac{\tau_1+\tau_2-\tau_3}{2}} (y_{13}^2)^{\frac{\tau_1+\tau_3-\tau_2}{2}} (y_{23}^2)^{\frac{\tau_2+\tau_3-\tau_1}{2}}}$$

$$\mathbf{n} = (n_1, n_2, n_3) \quad 0 \leq n_1 \leq \min\{s_2 - n_3, s_3 - n_2\} \quad 0 \leq n_2 \leq \min\{s_1 - n_3, s_3\} \quad 0 \leq n_3 \leq \min\{s_1, s_2\}$$

Six conformally covariant building blocks:

$$(Y_i)_\mu = \frac{(y_i - y_{i+1})_\mu}{(y_i - y_{i+1})^2} - \frac{(y_i - y_{i+2})_\mu}{(y_i - y_{i+2})^2}, \quad (H_i)_{\mu\nu} = \frac{1}{(y_{i+1} - y_{i+2})^2} \left( \delta_{\mu\nu} + 2 \frac{(y_{i+1} - y_{i+2})_\mu (y_{i+1} - y_{i+2})_\nu}{(y_{i+1} - y_{i+2})^2} \right)$$

$$i = 1, 2, 3 \quad i \cong i+3$$

# General Approach

## Projecting out the shadow

$$\tilde{\mathcal{O}}(y) = \kappa \int d^d y' \frac{I(y-y')}{[(y-y')^2]^{d-\Delta}} \mathcal{O}(y')$$

### Spectral representation

$$W_{\Delta,s} = \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + (\Delta - \frac{d}{2})^2} W_{\frac{d}{2} + i\nu, s} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + (\Delta - \frac{d}{2})^2} \left[ W_{\frac{d}{2} + i\nu, s} + W_{\frac{d}{2} - i\nu, s} \right]$$

pole in spectral function at:  $\frac{d}{2} + i\nu = \Delta$

$$\int d^d y \langle \langle \mathcal{O}_{\Delta_1, s_1} \mathcal{O}_{\Delta_2, s_2} \mathcal{O}_{\frac{d}{2} + i\nu, s}(y) \rangle \rangle \langle \langle \tilde{\mathcal{O}}_{\frac{d}{2} + i\nu, s}(y) \mathcal{O}_{\Delta_3, s_3} \mathcal{O}_{\Delta_4, s_4} \rangle \rangle$$

Gives rise to useful representation of CPWE:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \sum_s \int_{-\infty}^{\infty} d\nu c_s(\nu) W_{\frac{d}{2} + i\nu, s}$$

Contribution from some primary operator  $\mathcal{O}$  :

$$c_s(\nu) \sim \frac{c_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2} c_{\mathcal{O}\mathcal{O}_3\mathcal{O}_4}}{\nu^2 + (\Delta - \frac{d}{2})^2} + \dots$$

**Bulk side.**

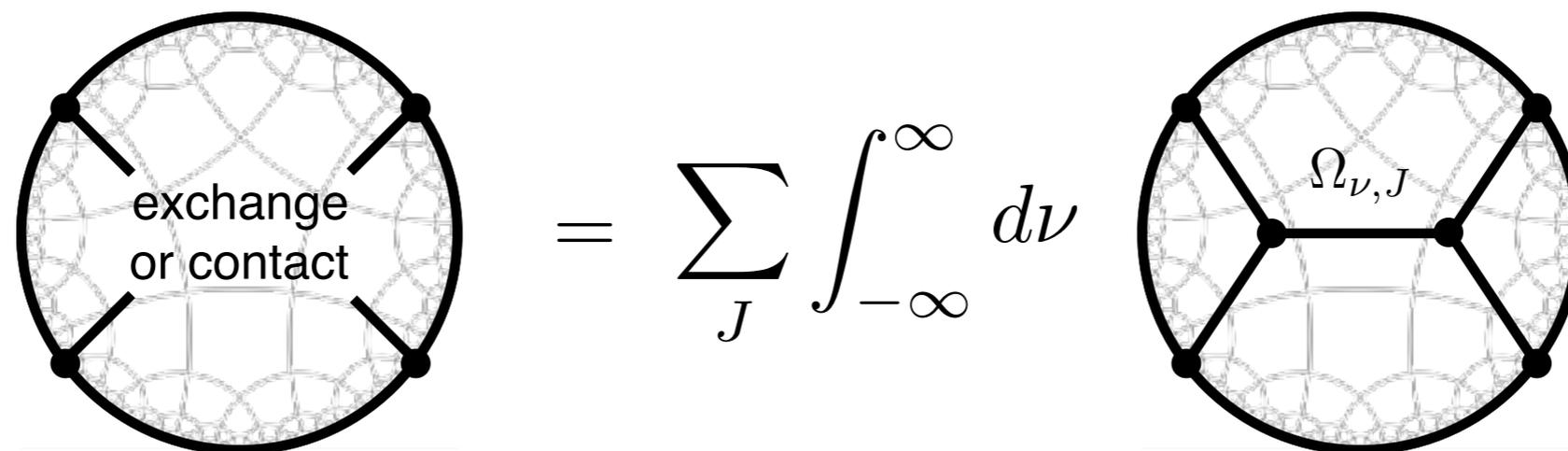
# General Approach

## Harmonic function decomposition of Witten diagrams

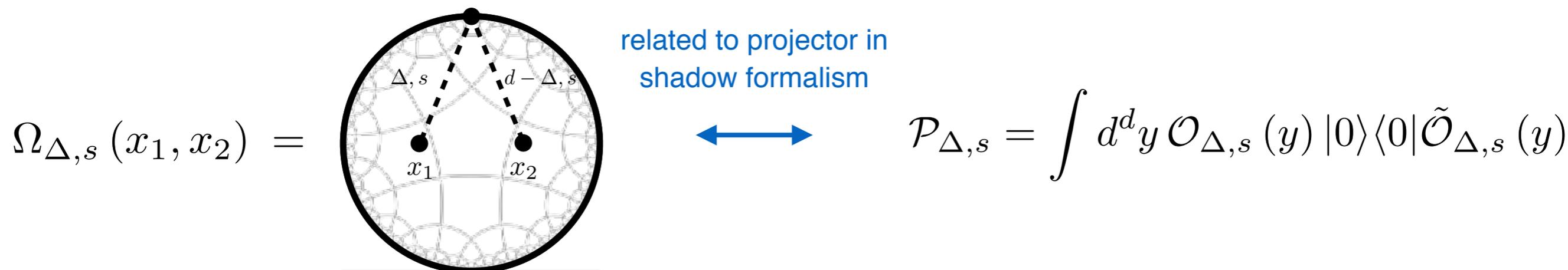
$$\square \Omega_{\Delta,s}(x_1, x_2) \propto \Omega_{\Delta,s}(x_1, x_2), \quad \nabla \cdot \Omega_{\Delta,s} = 0, \quad (g \cdot \Omega_{\Delta,s}) = 0$$

divergenceless traceless

Furnish complete basis of bi-tensors in AdS



Factorisation of harmonic functions:



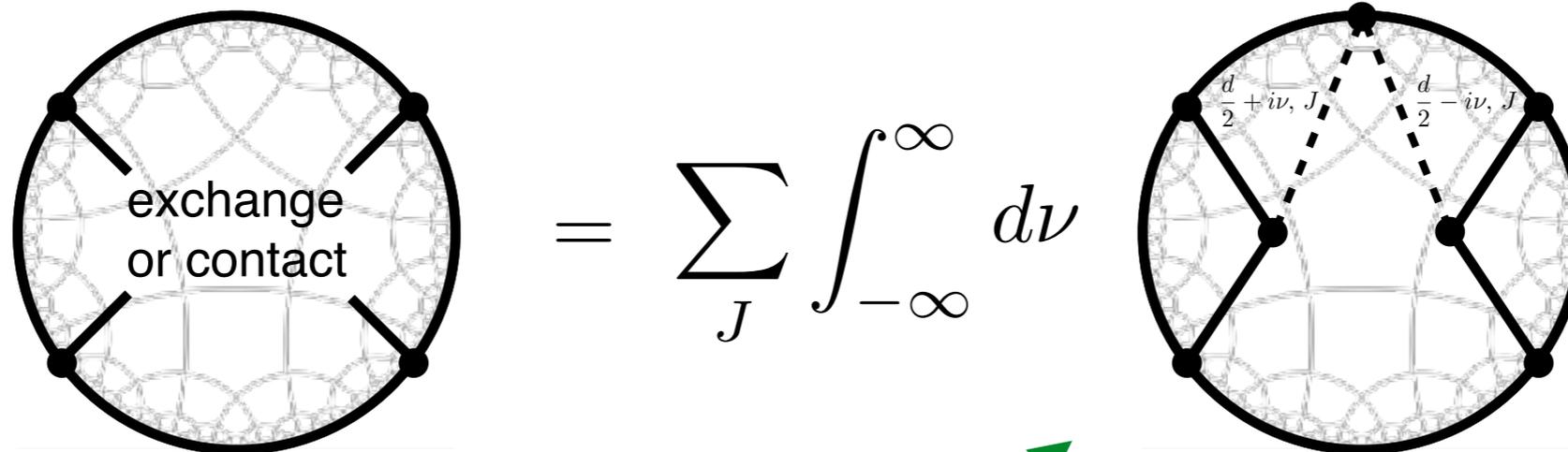
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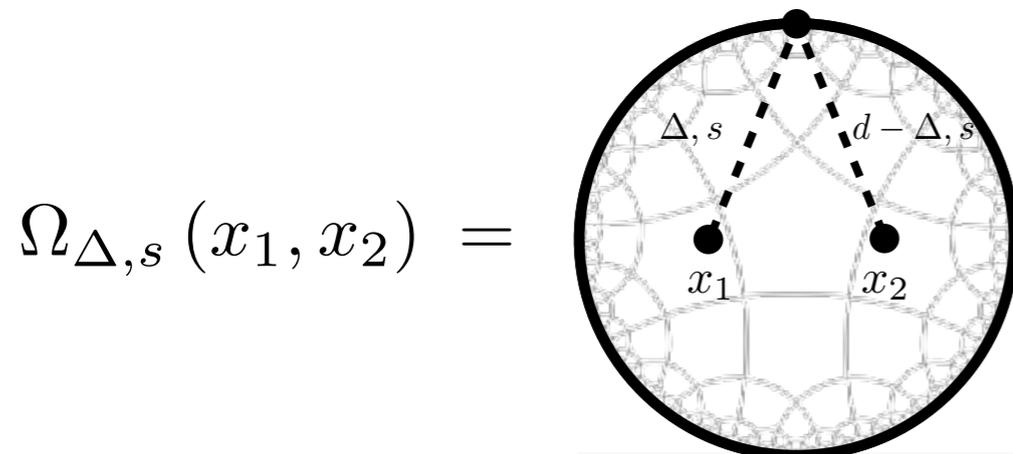
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Factorisation of harmonic functions:



related to projector in shadow formalism

$$\mathcal{P}_{\Delta,s} = \int d^d y \mathcal{O}_{\Delta,s}(y) |0\rangle \langle 0| \tilde{\mathcal{O}}_{\Delta,s}(y)$$

**Factorise into 3pt Witten diagrams!**

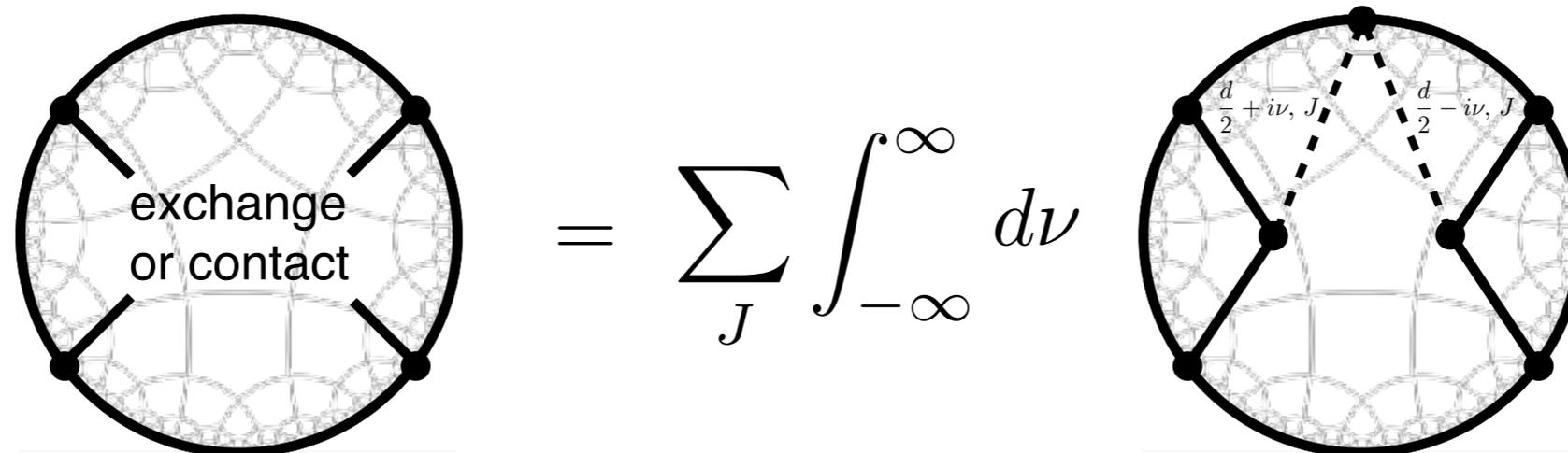
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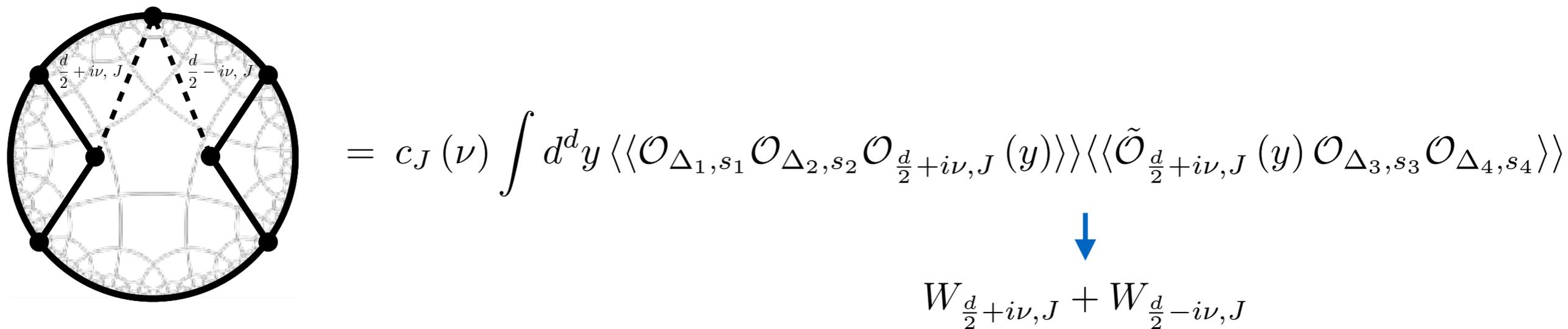
divergenceless traceless

Furnish complete basis of bi-tensors in AdS



$$\text{exchange or contact} = \sum_J \int_{-\infty}^{\infty} d\nu$$

## Evaluate 3pt Witten diagrams:



$$= c_J(\nu) \int d^d y \langle\langle \mathcal{O}_{\Delta_1, s_1} \mathcal{O}_{\Delta_2, s_2} \mathcal{O}_{\frac{d}{2} + i\nu, J}(y) \rangle\rangle \langle\langle \tilde{\mathcal{O}}_{\frac{d}{2} + i\nu, J}(y) \mathcal{O}_{\Delta_3, s_3} \mathcal{O}_{\Delta_4, s_4} \rangle\rangle$$

↓

$$W_{\frac{d}{2} + i\nu, J} + W_{\frac{d}{2} - i\nu, J}$$

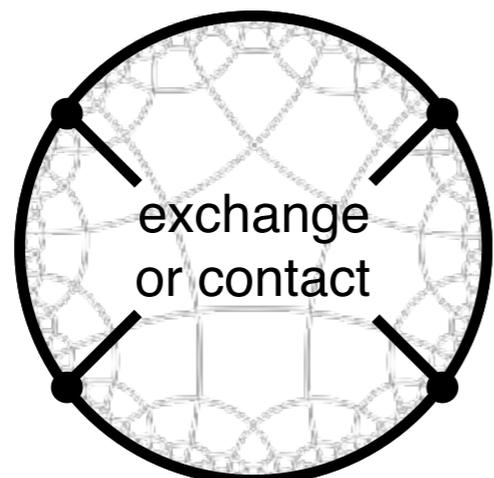
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## Harmonic function decomposition of Witten diagrams

$$\square \Omega_{\Delta,s}(x_1, x_2) \propto \Omega_{\Delta,s}(x_1, x_2), \quad \nabla \cdot \Omega_{\Delta,s} = 0, \quad (g \cdot \Omega_{\Delta,s}) = 0$$

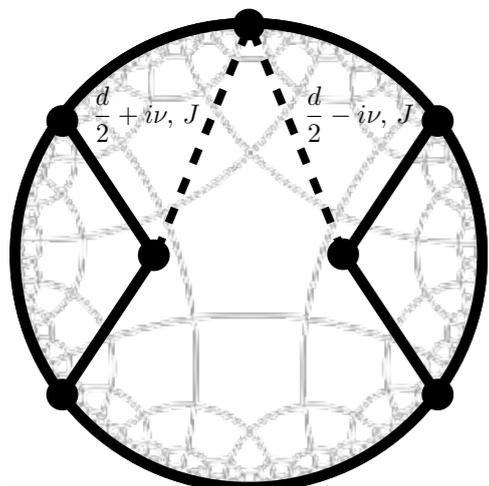
divergenceless traceless

Furnish complete basis of bi-tensors in AdS



$$= \sum_J \int_{-\infty}^{\infty} d\nu c_J(\nu) W_{\frac{d}{2}+i\nu, J}$$

## Evaluate 3pt Witten diagrams:



$$= c_J(\nu) \int d^d y \langle\langle \mathcal{O}_{\Delta_1, s_1} \mathcal{O}_{\Delta_2, s_2} \mathcal{O}_{\frac{d}{2}+i\nu, J}(y) \rangle\rangle \langle\langle \tilde{\mathcal{O}}_{\frac{d}{2}+i\nu, J}(y) \mathcal{O}_{\Delta_3, s_3} \mathcal{O}_{\Delta_4, s_4} \rangle\rangle$$

↓

$$W_{\frac{d}{2}+i\nu, J} + W_{\frac{d}{2}-i\nu, J}$$

# Outline

- General approach

- Spinning 3pt

- Spinning 4pt exchange diagrams

# Spinning 3pt Diagrams

Want to evaluate:

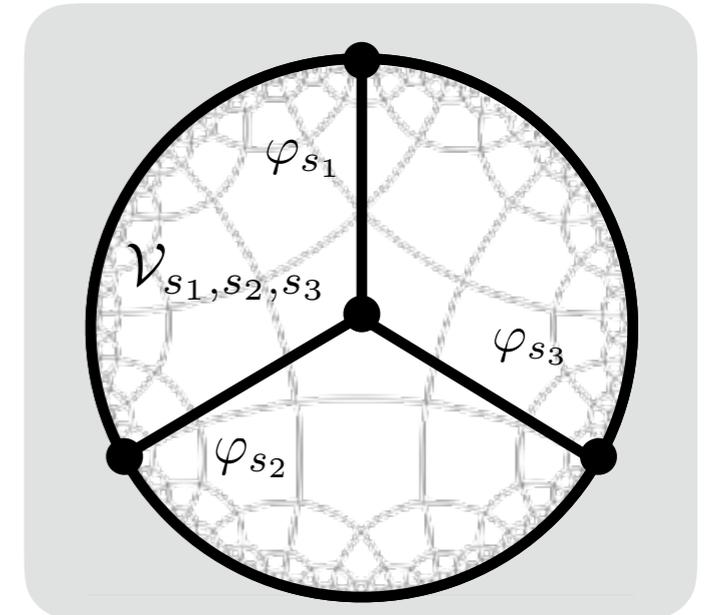
Basis of bulk cubic couplings

$$\mathcal{V}_{s_1, s_2, s_3} = \sum_{n_i} g_{s_1, s_2, s_3}^{n_1, n_2, n_3} \mathcal{V}_{s_1, s_2, s_3}^{n_1, n_2, n_3}$$

$$0 \leq n_1 \leq \min\{s_2 - n_3, s_3 - n_2\} \quad 0 \leq n_2 \leq \min\{s_1 - n_3, s_3\} \quad 0 \leq n_3 \leq \min\{s_1, s_2\}$$

with basis elements:

$$\begin{aligned} \mathcal{V}_{s_1, s_2, s_3}^{n_1, n_2, n_3} &= g^{\mu(n_3)\nu(n_3)} g^{\nu(n_1)\sigma(n_1)} g^{\sigma(n_2)\mu(n_2)} \nabla^{k(\bar{s}_3)} \varphi_{\mu(n_2+n_3)p(\bar{s}_1)} \\ &\quad \times \nabla^{p(\bar{s}_1)} \varphi_{\nu(n_3+n_1)q(\bar{s}_2)} \nabla^{q(\bar{s}_2)} \varphi_{\sigma(n_1+n_2)k(\bar{s}_3)} \end{aligned}$$



$$\varphi_{s_i} \text{ of mass } m_i^2 R^2 = \Delta_i (\Delta_i - d) - s_i$$

Notation

$$\bar{s}_i = s_i - (n_{i-1} + n_{i+1}) \quad \varphi_{\mu(s)} = \varphi_{\mu_1 \dots \mu_s} \quad g^{\mu(n)\nu(n)} = g^{\mu_1 \nu_1} \dots g^{\mu_n \nu_n} \quad \nabla^{\mu(n)} = \nabla^{\mu_1} \dots \nabla^{\mu_n}$$

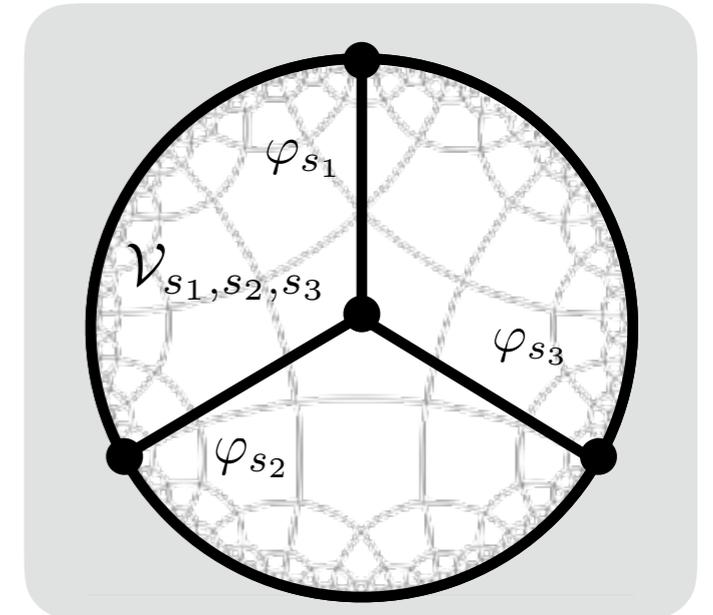
# Spinning 3pt Diagrams

Want to evaluate:

Basis of bulk cubic couplings

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$$\varphi_{s_i} \text{ of mass } m_i^2 R^2 = \Delta_i (\Delta_i - d) - s_i$$

with basis elements: (6 basic contractions)

$$\begin{aligned} \mathcal{V}_{s_1, s_2, s_3}^{n_1, n_2, n_3} &= g^{\mu(n_3)\nu(n_3)} g^{\nu(n_1)\sigma(n_1)} g^{\sigma(n_2)\mu(n_2)} \nabla^{k(\bar{s}_3)} \varphi_{\mu(n_2+n_3)p(\bar{s}_1)} \\ &\quad \times \nabla^{p(\bar{s}_1)} \varphi_{\nu(n_1+n_3)q(\bar{s}_2)} \nabla^{q(\bar{s}_2)} \varphi_{\sigma(n_1+n_2)k(\bar{s}_3)} \end{aligned}$$

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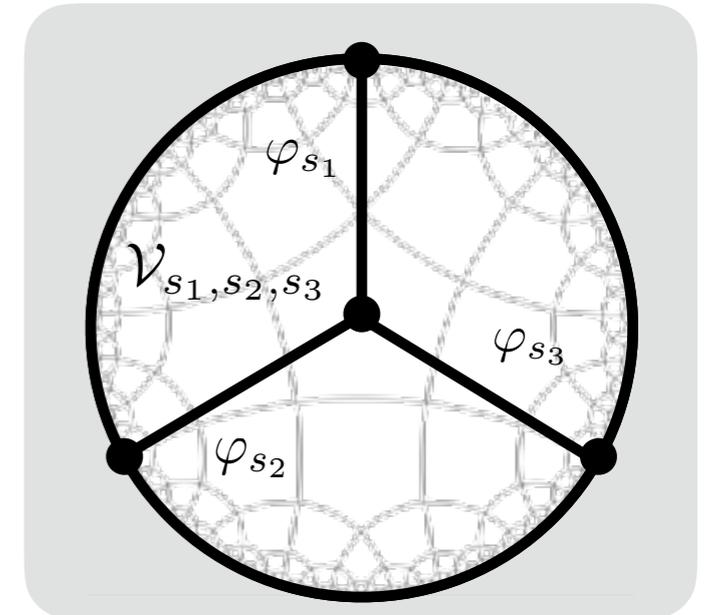
# Spinning 3pt Diagrams

Want to evaluate:

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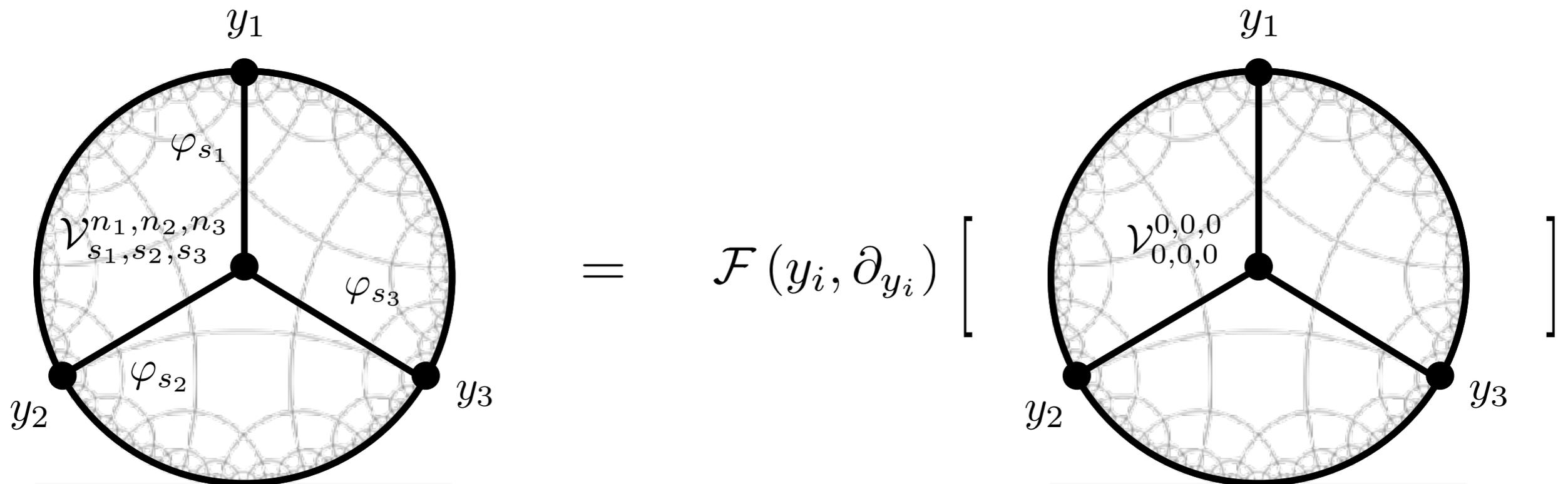
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# Spinning 3pt Diagrams

## Integral over AdS

**Trick:** Reduce integral over AdS to its scalar seed

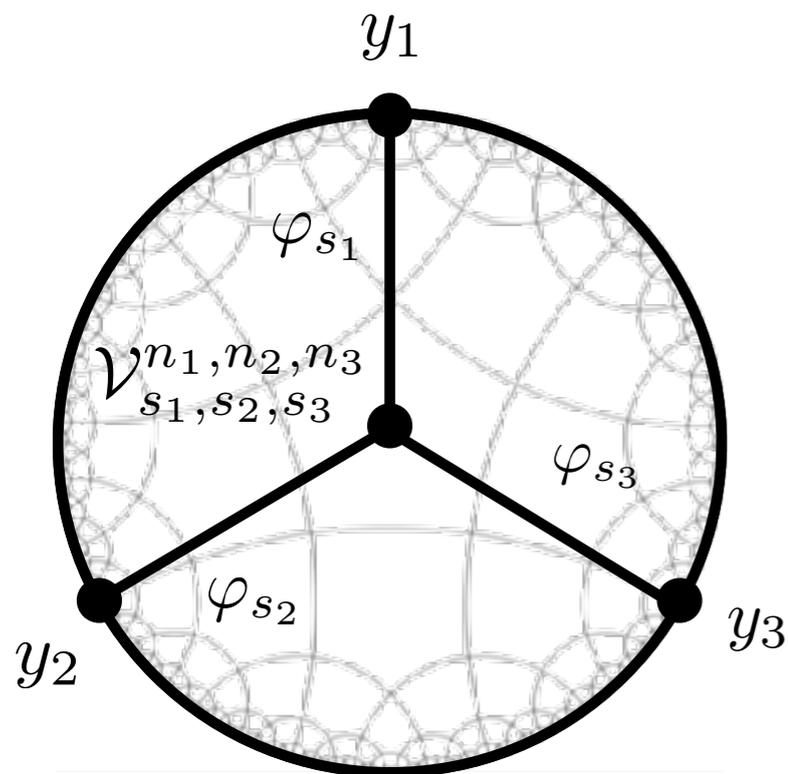


Well known how to compute  
[Mück et al.; Freedman et al. '98]

# Spinning 3pt Diagrams

## Integral over AdS

**Result:**



$$= \sum_{m_i} C_{m_1, m_2, m_3}^{n_1, n_2, n_3} \langle \langle \mathcal{O}_{\Delta_1, s_1}(y_1) \mathcal{O}_{\Delta_2, s_2}(y_2) \mathcal{O}_{\Delta_3, s_3}(y_3) \rangle \rangle^{(\mathbf{m})}$$

**Recall:** Basis of 3pt spinning conformal structures

$$\tau_i = \Delta_i - s_i$$

$$\langle \langle \mathcal{O}_{\Delta_1, s_1}(y_1) \mathcal{O}_{\Delta_2, s_2}(y_2) \mathcal{O}_{\Delta_3, s_3}(y_3) \rangle \rangle^{(\mathbf{n})} = \frac{Y_1^{s_1 - n_2 - n_3} Y_2^{s_2 - n_1 - n_3} Y_3^{s_3 - n_1 - n_2} H_1^{n_1} H_2^{n_2} H_3^{n_3}}{(y_{12}^2)^{\frac{\tau_1 + \tau_2 - \tau_3}{2}} (y_{13}^2)^{\frac{\tau_1 + \tau_3 - \tau_2}{2}} (y_{23}^2)^{\frac{\tau_2 + \tau_3 - \tau_1}{2}}}$$

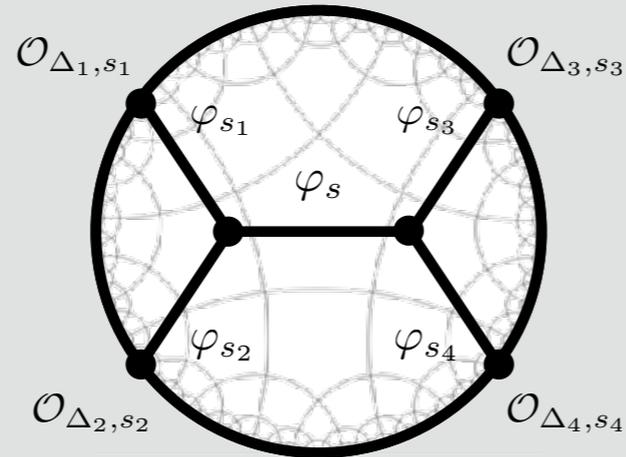
# Outline

- General approach
- Spinning 3pt
- Spinning 4pt exchange diagrams

# Spinning Exchange Diagrams

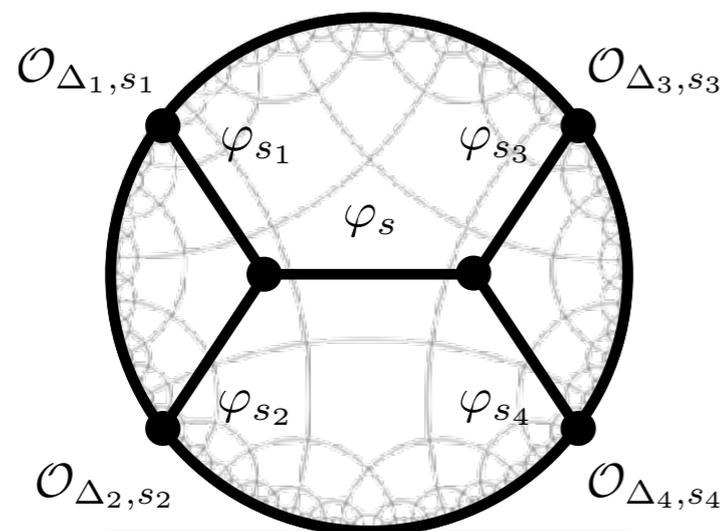
**Goal:**

Decompose

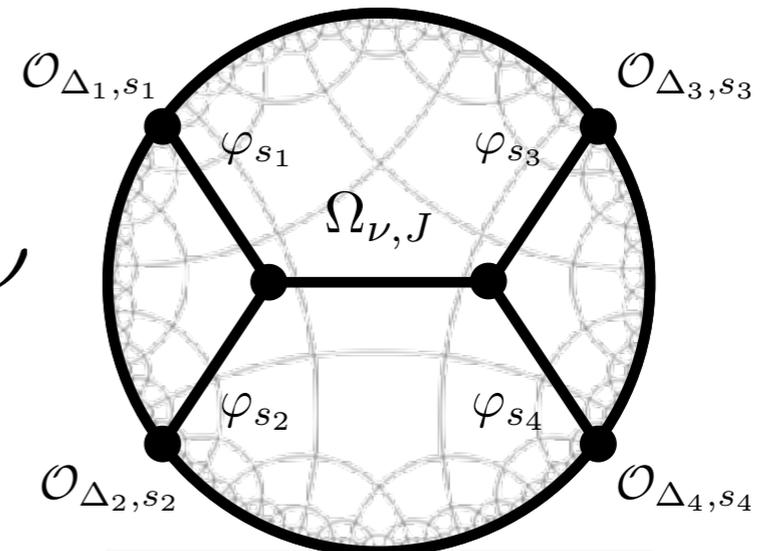


into Conformal Partial Waves

**First step:** Decomposition into harmonic functions  $\Omega_{\nu, J}$



$$= \sum_J \int_{-\infty}^{\infty} d\nu$$



# Spinning Exchange Diagrams

**First step:** Harmonic function decomposition

$$\begin{array}{c}
 \text{Left Diagram} \\
 \text{Right Diagram}
 \end{array}
 = \sum_J \int_{-\infty}^{\infty} d\nu$$

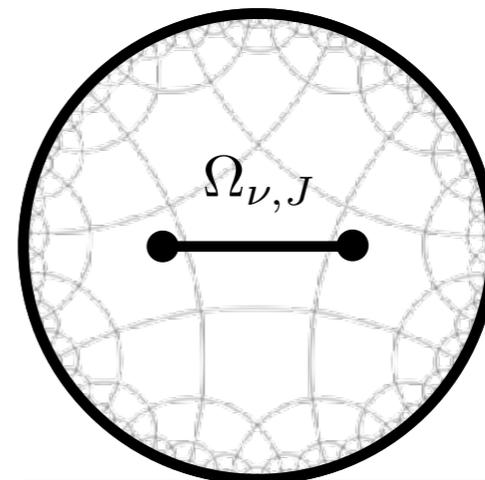
**Bulk-to-bulk propagator**  $\Pi_s$

$$m_s^2 R^2 = \Delta(\Delta - d) - s$$

$$\varphi_s(x) = \int_{\text{AdS}} d^{d+1}x' \Pi_s(x, x') \cdot J_s(x'), \quad (\square_1 - m_s^2 + \dots) \Pi_s(x_1, x_2) = -\delta^{d+1}(x_1, x_2)$$

Express in basis of harmonic functions  $\Omega_{\nu, J}$ :

$$\Pi_s(x_1, x_2) = \sum_{J=0}^s \int_{-\infty}^{\infty} d\nu f_J(\nu)$$



# Spinning Exchange Diagrams

**First step:** Harmonic function decomposition

$$= \sum_{J=0}^s \int_{-\infty}^{\infty} d\nu$$

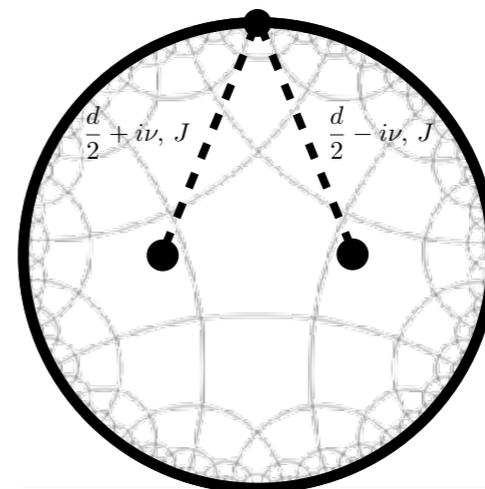
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# Spinning Exchange Diagrams

**First step:** Harmonic function decomposition

The diagram illustrates the harmonic function decomposition of a four-point exchange diagram. On the left, a circular disk contains four external operators  $\mathcal{O}_{\Delta_1, s_1}$ ,  $\mathcal{O}_{\Delta_2, s_2}$ ,  $\mathcal{O}_{\Delta_3, s_3}$ , and  $\mathcal{O}_{\Delta_4, s_4}$  connected by solid lines. The internal region is divided into four triangles with harmonic functions  $\varphi_{s_1}$ ,  $\varphi_{s_2}$ ,  $\varphi_{s_3}$ , and  $\varphi_{s_4}$ , and a central region with  $\varphi_s$ . This is equal to a sum over  $s$  from  $J=0$  to  $\infty$  of an integral over  $dt$  from  $-\infty$  to  $\infty$  of a similar diagram. In the right-hand diagram, a dashed line connects the two internal vertices, and a red oval highlights the entire right-hand side of the equation.

$$= \sum_{J=0}^s \int_{-\infty}^{\infty} dt$$

**Second step:** Evaluate the 3pt Witten diagrams

The diagram shows the evaluation of a three-point Witten diagram. On the left, a circular disk has three external operators  $\mathcal{O}_{\Delta_1, s_1}$ ,  $\mathcal{O}_{\Delta_2, s_2}$ , and  $\mathcal{O}_{\Delta_3, s_3}$  at positions  $y_1$ ,  $y_2$ , and  $y_3$  respectively. The internal region is divided into three triangles with harmonic functions  $\varphi_{s_1}$ ,  $\varphi_{s_2}$ , and  $\varphi_{s_3}$ . The central region is labeled with  $\mathcal{V}_{s_1, s_2, s_3}^{n_1, n_2, n_3}$ . This is equal to a sum over  $m_i$  of a correlator of three operators:  $\langle\langle \mathcal{O}_{\Delta_1, s_1}(y_1) \mathcal{O}_{\Delta_2, s_2}(y_2) \mathcal{O}_{\Delta_3, s_3}(y_3) \rangle\rangle^{(\mathbf{m})}$ .

$$= \sum_{m_i} C_{m_1, m_2, m_3}^{n_1, n_2, n_3} \langle\langle \mathcal{O}_{\Delta_1, s_1}(y_1) \mathcal{O}_{\Delta_2, s_2}(y_2) \mathcal{O}_{\Delta_3, s_3}(y_3) \rangle\rangle^{(\mathbf{m})}$$

# Spinning Exchange Diagrams

**First step:** Harmonic function decomposition

$$= \sum_{J=0}^s \int_{-\infty}^{\infty} d\nu$$

$$\int d^d y \langle \langle \mathcal{O}_{\Delta_1, s_1} \mathcal{O}_{\Delta_2, s_2} \mathcal{O}_{\frac{d}{2} + i\nu, J}(y) \rangle \rangle^{(\mathbf{m})} \langle \langle \tilde{\mathcal{O}}_{\frac{d}{2} + i\nu, J}(y) \mathcal{O}_{\Delta_3, s_3} \mathcal{O}_{\Delta_4, s_4} \rangle \rangle^{(\mathbf{n})}$$

**Second step:** Evaluate the 3pt Witten diagrams

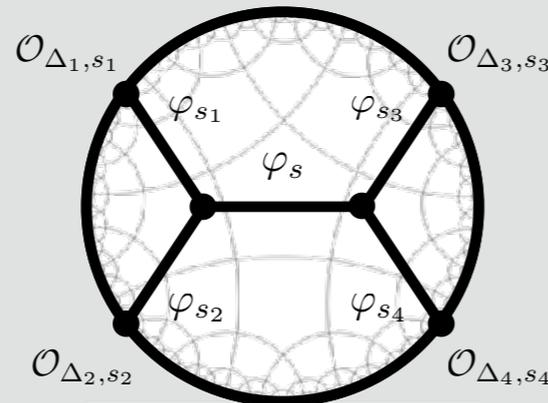
$$= \sum_{m_i} C_{m_1, m_2, m_3}^{n_1, n_2, n_3} \langle \langle \mathcal{O}_{\Delta_1, s_1}(y_1) \mathcal{O}_{\Delta_2, s_2}(y_2) \mathcal{O}_{\Delta_3, s_3}(y_3) \rangle \rangle^{(\mathbf{m})}$$

To determine the CPWE, recall:

$$W_{\Delta, s}^{\mathbf{m}, \mathbf{n}} + W_{d-\Delta, s}^{\mathbf{m}, \mathbf{n}} = \kappa \int d^d y \langle \langle \mathcal{O}_{\Delta_1, s_1} \mathcal{O}_{\Delta_2, s_2} \mathcal{O}(y) \rangle \rangle^{(\mathbf{m})} \langle \langle \tilde{\mathcal{O}}(y) \mathcal{O}_{\Delta_3, s_3} \mathcal{O}_{\Delta_4, s_4} \rangle \rangle^{(\mathbf{n})}$$

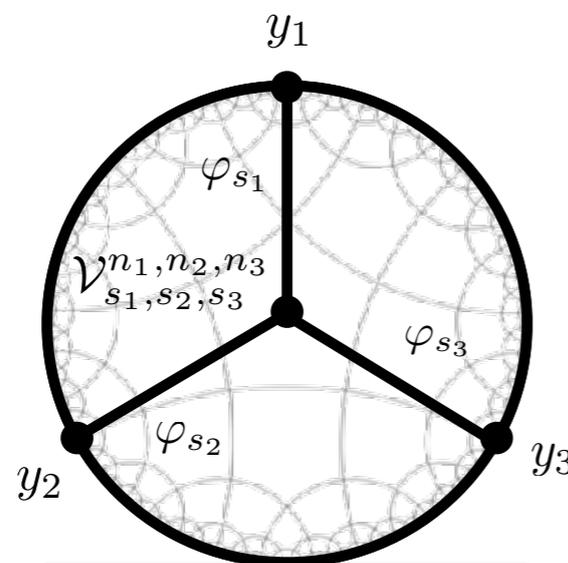
# Spinning Exchange Diagrams

**Final step:** Conformal Partial Wave Expansion



$$= \sum_{J=0}^s \int_{-\infty}^{\infty} d\nu c_J^{\mathbf{m}, \mathbf{n}}(\nu) W_{\frac{d}{2} + i\nu, J}^{\mathbf{m}, \mathbf{n}}$$

**Second step:** Evaluate the 3pt Witten diagrams

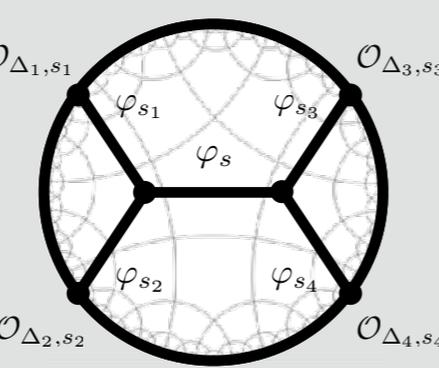


$$= \sum_{m_i} C_{m_1, m_2, m_3}^{n_1, n_2, n_3} \langle\langle \mathcal{O}_{\Delta_1, s_1}(y_1) \mathcal{O}_{\Delta_2, s_2}(y_2) \mathcal{O}_{\Delta_3, s_3}(y_3) \rangle\rangle^{(\mathbf{m})}$$

To determine the CPWE, recall:

$$W_{\Delta, s}^{\mathbf{m}, \mathbf{n}} + W_{d-\Delta, s}^{\mathbf{m}, \mathbf{n}} = \kappa \int d^d y \langle\langle \mathcal{O}_{\Delta_1, s_1} \mathcal{O}_{\Delta_2, s_2} \mathcal{O}(y) \rangle\rangle^{(\mathbf{m})} \langle\langle \tilde{\mathcal{O}}(y) \mathcal{O}_{\Delta_3, s_3} \mathcal{O}_{\Delta_4, s_4} \rangle\rangle^{(\mathbf{n})}$$

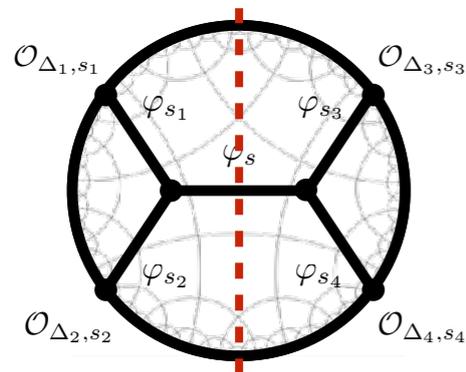
# Spinning Exchange Diagrams



$$= \sum_{J=0}^s \int_{-\infty}^{\infty} d\nu c_J^{\mathbf{m}, \mathbf{n}}(\nu) W_{\frac{d}{2} + i\nu, J}^{\mathbf{m}, \mathbf{n}}$$

## Two types of contributions:

1. Single trace operator  $\mathcal{O}_{\Delta, s}$  dual to exchanged single-particle state  $\varphi_s$



non-coincident  
bulk points

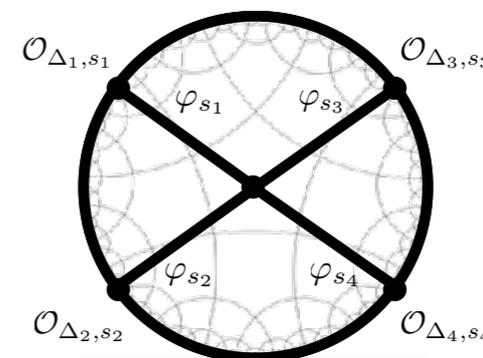
$$c_s^{\mathbf{m}, \mathbf{n}}(\nu) \sim \frac{c_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2}^{\mathbf{m}} c_{\mathcal{O}\mathcal{O}_3\mathcal{O}_4}^{\mathbf{n}}}{\nu^2 + \left(\Delta - \frac{d}{2}\right)^2} + \dots$$

2. Double-trace operators  $[\mathcal{O}_{\Delta_1, s_1} \mathcal{O}_{\Delta_2, s_2}]_J$  and  $[\mathcal{O}_{\Delta_3, s_3} \mathcal{O}_{\Delta_4, s_4}]_J$  of spin  $J = 0, \dots, s$

Scaling dimensions:

$$\Delta_1 + \Delta_2 - (s_1 + s_2) + J + 2n, \quad \Delta_3 + \Delta_4 - (s_3 + s_4) + J + 2n$$

$$n = 0, 1, 2, 3 \dots$$



arise from  
contact terms  
in the exchange

# Summary

- New tools to evaluate 3pt and 4pt Spinning Witten diagrams
- Spells out the map between OPE coeffs. and bulk couplings of totally symmetric fields
- Given any  $\text{CFT}_d$ , this provides holographic reconstruction of cubic and quartic interactions of totally symmetric fields on  $\text{AdS}_{d+1}$

## Some concrete applications so far:

- Computation of all 3pt diagrams and 4pt exchange diagrams in type A minimal higher-spin gauge theory on  $\text{AdS}_{d+1}$
- Holographic reconstruction of cubic and quartic interactions in type A higher-spin gauge theory