June , 2017

Triple parton scattering (TPS)

A.M. Snigirev,
D.V.Skobeltsyn Institute of Nuclear Physics
M.V.Lomonosov Moscow State University
Moscow, Russia, 119991

MPI Workshops (2008, 2010-2016):

P. Bartalini *et al.,* **arXiv:1003.4220 [hep-ep].**

P. Bartalini *et al.,* **arXiv:1111.0469 [hep-ph].**

H. Abramowicz *et al.,* **arXiv:1306.5413 [hep-ph].**

S. Bansal *et al.,* **arXiv:1410.6664 [hep-ph].**

R. Astalos *et al.,* **arXiv:1506.05829 [hep-ph].**

mpi@lhc 2015, H. Jung *et al.,* **DESY-PROC-2016-01.**

mpi@lhc 2016 (Mexico), https://indico.nucleares.unam.mx/event/1100/

PARTON MODEL

Elastic scattering : electron — proton

——–> proton (hadron) is NOT point-like

Deep inelastic scattering : electron — proton

——–> proton (hadron) consists of point-like particles-partons

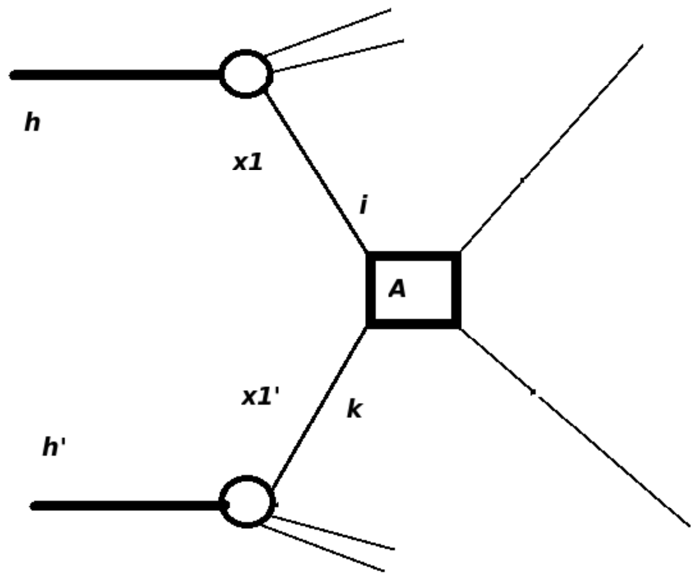
---

$$\text{Cross section (hadron)} \;=\; \Sigma \;\text{cross section (parton)} \times \text{weights}$$

Weights — probabilities in the system of infinite momentum

(Bjorken, Feynman)

IN QCD weights depend on Q of hard processes
(SCALING VIOLATION, improved PM)



$$\sigma^A_{\mathrm{SPS}} = \sum_{i,k} \int D^i_h(x_1;Q_1^2)\hat{\sigma}^A_{ik}(x_1,x_1')D^k_{h'}(x_1';Q_1^2)dx_1dx_1'$$

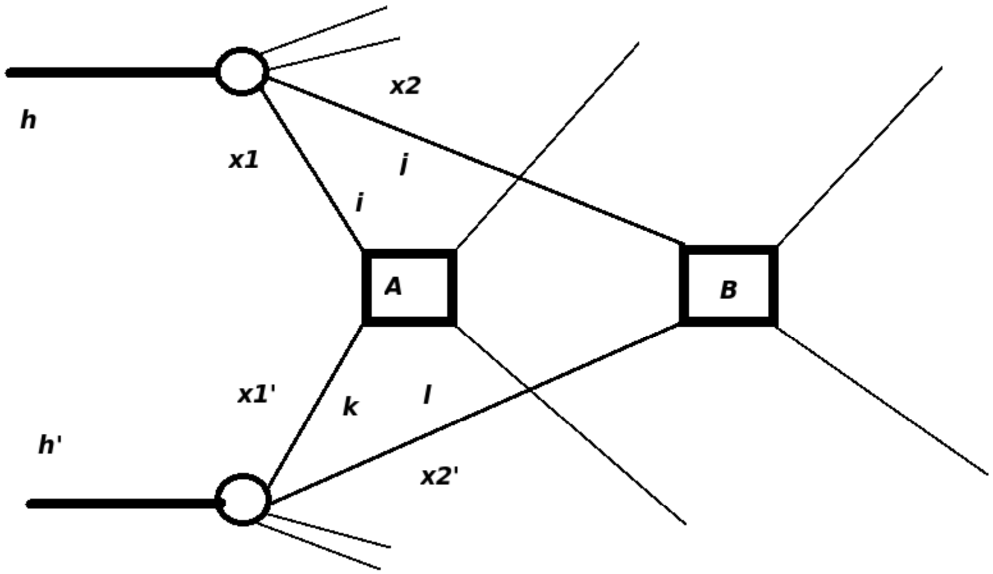
Scaling violation (dependence on Q) from
DGLAP (*Dokshitzer-Gribov-Lipatov-Altarelli-Parisi*) equations:

$$\frac{dD_i^j(x,t)}{dt} = \sum_{j'} \int_x^1 \frac{dx'}{x'} D_i^{j'}(x',t) P_{j'\to j}\left(\frac{x}{x'}\right)$$

$$t = \frac{1}{2\pi b} \ln\left[1 + \frac{g^2(\mu^2)}{4\pi} b \ln\left(\frac{Q^2}{\mu^2}\right)\right] = \frac{1}{2\pi b} \ln\left[\frac{\ln\left(\frac{Q^2}{\Lambda^2_{QCD}}\right)}{\ln\left(\frac{\mu^2}{\Lambda^2_{QCD}}\right)}\right], \quad b = \frac{33 - 2n_f}{12\pi},$$

where $g(\mu^2)$ is the running coupling constant at the reference scale μ^2,
n_f is the number of active flavours,
Λ_{QCD} is the dimensional QCD parameter.

It is possible (BUT very rarely): hard double parton scattering (subprocesses A and B)



The inclusive cross section of a double parton scattering process in a hadron collision is written in the following form (with only the assumption of factorization of the two hard parton subprocesses A and B) *(Paver, Treleani,..., Blok,...., Diehl,...).*

$$\boldsymbol{\sigma^{AB}_{DPS} = \frac{m}{2} \sum_{i,j,k,l} \int \Gamma_{ij}(x_1, x_2; \mathrm{b}_1, \mathrm{b}_2; Q_1^2, Q_2^2) \hat{\sigma}^A_{ik}(x_1, x_1', Q_1^2) \hat{\sigma}^B_{jl}(x_2, x_2', Q_2^2)}$$

$$\boldsymbol{\times \Gamma_{kl}(x_1', x_2'; \mathrm{b}_1 - \mathrm{b}, \mathrm{b}_2 - \mathrm{b}; Q_1^2, Q_2^2) dx_1 dx_2 dx_1' dx_2' d^2b_1 d^2b_2 d^2b,}$$

where $\mathbf{b}$ is the impact parameter — the distance between centers of colliding (e.g., the beam and the target) hadrons in transverse plane.

$\Gamma_{ij}(x_1, x_2; \mathbf{b_1}, \mathbf{b_2}; Q_1^2, Q_2^2)$ are the double parton distribution functions, which depend on the longitudinal momentum fractions x_1 and x_2, and on the transverse position $\mathbf{b_1}$ and $\mathbf{b_2}$ of the two parton undergoing hard processes A and B at the scales Q_1 and Q_2.

$\hat{\sigma}^A_{ik}$ and $\hat{\sigma}^B_{jl}$ are the parton-level subprocess cross sections.

The factor $m/2$ appears due to the symmetry of the expression for interchanging parton species i and j. $m = 1$ if $A = B$, and $m = 2$ otherwise.

The double parton distribution functions $\Gamma_{ij}(x_1, x_2; \mathbf{b_1}, \mathbf{b_2}; Q_1^2, Q_2^2)$ are the main object of interest as concerns multiple parton interactions. In fact, these distributions contain all the information when probing the hadron in two different points simultaneously, through the hard processes A and B.

It is typically assumed that the double parton distribution functions may be decomposed in terms of longitudinal and transverse components as follows:

$$\Gamma_{ij}(x_1, x_2; \mathrm{b}_1, \mathrm{b}_2; Q_1^2, Q_2^2) = D_h^{ij}(x_1, x_2; Q_1^2, Q_2^2) f(\mathrm{b}_1) f(\mathrm{b}_2),$$

where $f(\mathbf{b_1})$ is supposed to be a universal function for all kinds of partons with the fixed normalization

$$\int f(\mathrm{b}_1) f(\mathrm{b}_1 - \mathrm{b}) d^2 b_1 d^2 b = \int T(\mathrm{b}) d^2 b = 1,$$

and

$$T(\mathbf{b}) = \int f(\mathbf{b_1}) f(\mathbf{b_1} - \mathbf{b}) d^2 b_1$$

is the overlap function (not calculated in pQCD).

If one makes the further assumption that the longitudinal components $D_h^{ij}(x_1, x_2; Q_1^2, Q_2^2)$ reduce to the product of two independent one parton distributions,

$$D_h^{ij}(x_1, x_2; Q_1^2, Q_2^2) = D_h^i(x_1; Q_1^2) D_h^j(x_2; Q_2^2),$$

the cross section of double parton scattering can be expressed in the simple form

$$\sigma_{\rm DPS}^{\rm AB} = \frac{m}{2} \frac{\sigma_{\rm SPS}^A \sigma_{\rm SPS}^B}{\sigma_{\rm eff}},$$

$$\pi R_{\rm eff}^2 = \sigma_{\rm eff} = [\int d^2b (T(\mathrm{b}))^2]^{-1}$$

is the effective interaction transverse area (effective cross section). $R_{\rm eff}$ is an estimate of the size of the hadron.

The **momentum** (*instead of the mixed (momentum and coordinate)*) representation is more convenient sometimes:

$$\sigma^{AB}_{DPS} = \frac{m}{2} \sum_{i,j,k,l} \int \Gamma_{ij}(x_1, x_2; \mathrm{q}; Q_1^2, Q_2^2) \hat{\sigma}^A_{ik}(x_1, x_1^{'}) \hat{\sigma}^B_{jl}(x_2, x_2^{'})$$

$$\times \Gamma_{kl}(x_1^{'}, x_2^{'}; -\mathrm{q}; Q_1^2, Q_2^2) dx_1 dx_2 dx_1^{'} dx_2^{'} \frac{d^2 q}{(2\pi)^2}.$$

Here the transverse vector q is equal to the difference of the momenta of partons from the wave function of the colliding hadrons in the amplitude and the amplitude conjugated. Such dependence arises because the diffe - rence of parton transverse momenta within the parton pair is not conserved.

The main problems are

* to make the correct calculation of the two-parton functions $\Gamma_{ij}(x_1, x_2; \mathbf{q}; Q_1^2, Q_2^2)$ **WITHOUT simplifying factorization assumptions** (*which are not sufficiently justified and should be revised: (Blok, Dokshitzer, Frankfurt, Strikman; Diehl, Schafer; Gaunt, Stirling; Ryskin, Snigirev;...)*)

* to find (observe) longitudinal momentum parton correlations and deviation from the factorization form of DPS cross section.

These functions are available in the current literature only for $\mathbf{q} = 0$ in the collinear approximation. In this approximation the two-parton distribution functions, $\Gamma_{ij}(x_1, x_2; \mathbf{q} = 0; Q^2, Q^2) = D_h^{ij}(x_1, x_2; Q^2, Q^2)$ with the two hard scales set equal, satisfy the generalized DGLAP evolution equations *(Kirshner; Shelest, Snigirev, Zinovjev)*.

$$\frac{dD_i^{j_1 j_2}(x_1, x_2, t)}{dt} = \sum_{j_1{}'} \int_{x_1}^{1-x_2} \frac{dx_1{}'}{x_1{}'} D_i^{j_1{}' j_2}(x_1{}', x_2, t) P_{j_1{}' \to j_1}\left(\frac{x_1}{x_1{}'}\right)$$

$$+ \sum_{j_2{}'} \int_{x_2}^{1-x_1} \frac{dx_2{}'}{x_2{}'} D_i^{j_1 j_2{}'}(x_1, x_2{}', t) P_{j_2{}' \to j_2}\left(\frac{x_2}{x_2{}'}\right)$$

$$+ \sum_{j'} D_i^{j'}(x_1 + x_2, t) \frac{1}{x_1 + x_2} P_{j' \to j_1 j_2}\left(\frac{x_1}{x_1 + x_2}\right)$$

The solutions of the generalized DGLAP evolution equations with the given initial conditions at the reference scales $\mu^2(t=0)$ may be written in the form:

$$D_h^{j_1 j_2}(x_1, x_2, t) \; = \; D_{h1}^{j_1 j_2}(x_1, x_2, t) \; + D_{h(QCD)}^{j_1 j_2}(x_1, x_2, t),$$

where

$$D_{h1}^{j_1 j_2}(x_1, x_2, t) =$$

$$\sum_{j_1' j_2'} \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} \; D_h^{j_1' j_2'}(z_1, z_2, 0) D_{j_1'}^{j_1}(\frac{x_1}{z_1}, t) D_{j_2'}^{j_2}(\frac{x_2}{z_2}, t) \; ,$$

$$D_{h(QCD)}^{j_1 j_2}(x_1, x_2, t) =$$

$$\sum_{j' j_1' j_2'} \int_0^t dt' \int_{x_1}^{1-x_2} \frac{dz_1}{z_1} \int_{x_2}^{1-z_1} \frac{dz_2}{z_2} \; D_h^{j'}(z_1 + z_2, t')$$

$$\frac{1}{z_1 + z_2} P_{j' \to j_1' j_2'}(\frac{z_1}{z_1 + z_2}) D_{j_1'}^{j_1}(\frac{x_1}{z_1}, t - t') D_{j_2'}^{j_2}(\frac{x_2}{z_2}, t - t').$$

The first term is the solution of homogeneous evolution equation (independent evolution of two branches), where the input two-parton distribution is generally NOT known at the low scale $\mu(t = 0)$. For this non-perturbative two-parton function at low z_1, z_2 one may assume the factorization $D_h^{j_1' j_2'}(z_1, z_2, 0) \simeq D_h^{j_1'}(z_1, 0) D_h^{j_2'}(z_2, 0)$ neglecting the constraints due to momentum conservation ($z_1 + z_2 < 1$).

This leads to

$$\boldsymbol{D_{h1}^{ij}(x_1, x_2, t) \simeq D_h^i(x_1, t) D_h^j(x_2, t)}$$

the factorization hypothesis usually used in current estimations.

This MAIN result shows that if the two-parton distributions are factorized at some scale μ^2, then the evolution (second term) violates this factorization *inevitably* at any different scale ($Q^2 \neq \mu^2$), apart from the violation due to the kinematic correlations induced by the momentum conservation.

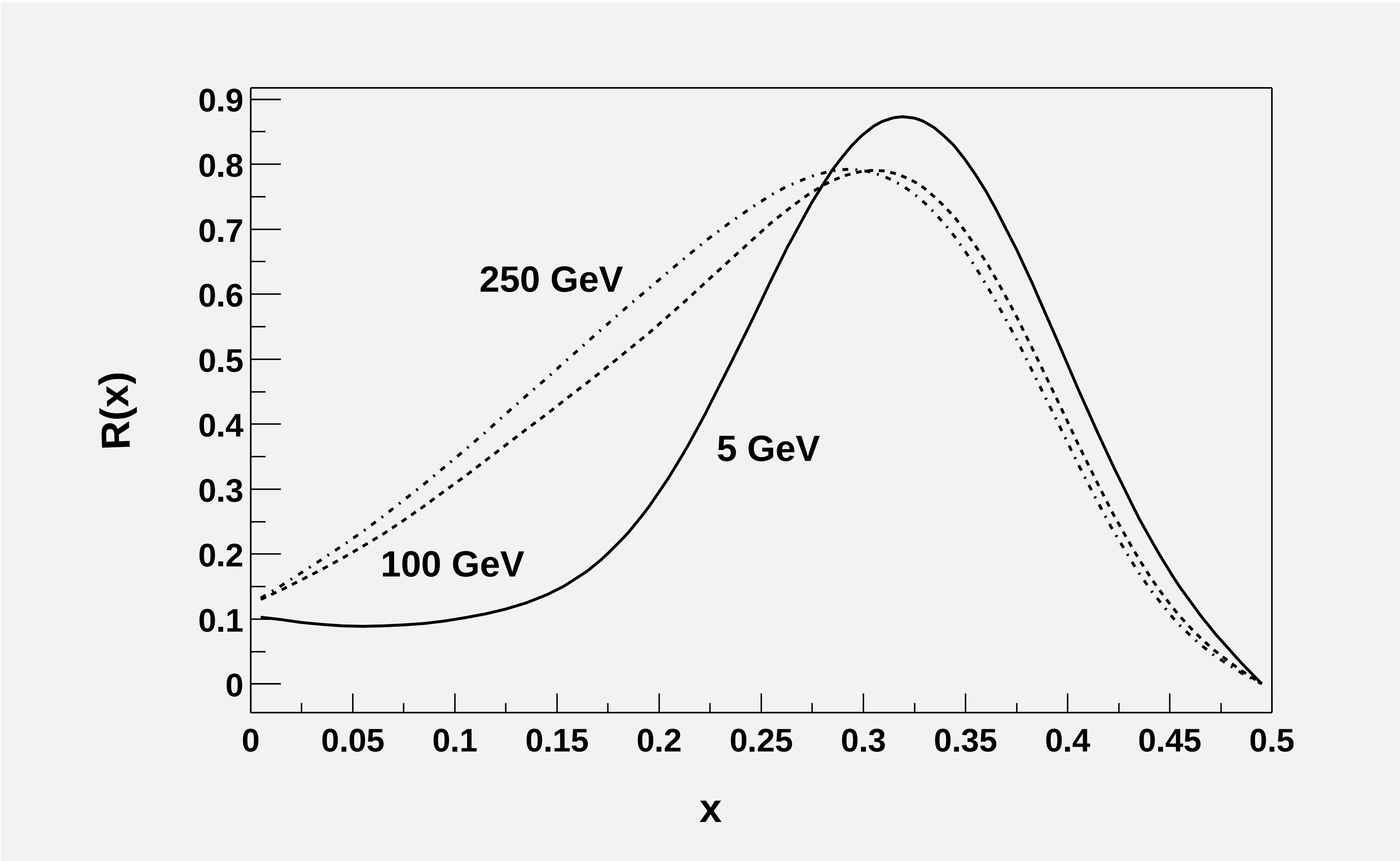
For a practical employment it is interesting to know the degree of this violation. We did *(Korotkikh, Snigirev)* it using the CTEQ fit for single distributions as an input. The nonperturbative initial conditions $D_h^j(x,0)$ are specified in a parametrized form at a fixed low-energy scale $Q_0 = \mu = 1.3$ GeV. The particular function forms and the value of Q_0 are not crucial for the CTEQ global analysis at the flexible enough parametrization, which reads

$$xD_p^j(x,0) \;=\; A_0^j x^{A_1^j}(1-x)^{A_2^j} e^{A_3^j x}(1+e^{A_4^j}x)^{A_5^j}.$$

The independent parameters $A_0^j,\; A_1^j,\; A_2^j,\; A_3^j,\; A_4^j,\; A_5^j$ for parton flavour combinations $u_v \equiv u - \bar{u}$, $d_v \equiv d - \bar{d}$, g and $\bar{u} + \bar{d}$ are given in Appendix A of work: *J.Pamplin, et al., JHEP 0207 (2002) 012.*

The results of numerical calculations are presented in Fig. for the ratio:

$$R(x,t) \;=\; (D^{gg}_{p(QCD)}(x_1,x_2,t)/D^g_p(x_1,t)D^g_p(x_2,t)(1-x_1-x_2)^2)|_{x_1=x_2=x}.$$



R(x)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
250 GeV
5 GeV
100 GeV
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
x

The evolution effects are getting larger with increasing hard scales. The numerical estimations by integrating **directly** the evolution equations (*Gaunt, Stirling; Diehl, Kasemets, Keane*) confirm also this conclusion.

The particular solutions of non-homogeneous equations contribute to the inclusive cross section of DPS with a **larger weight** (different effective cross section *(Cattaruzza,Del Fabbro,Treleani; Ryskin, Snigirev; Blok, Dokshitzer, Frankfurt, Strikman; Gaunt, Stirling)*) as compared to the solutions of homogeneous equations (**the "traditional" factorization component**).

The latter solutions are usually approximated by a factorized form if the initial nonperturbative correlations are absent. These initial correlation conditions are *a priori* unknown yet not quite arbitrary as they obey the nontrivial sum rules which are imposed upon the evolution equations. The problem of specifying the initial correlation conditions for the evolution equations, which would obey exactly these **sum rules** and have the **correct asymptotic behavior near the kinematical boundaries**, has been extensively studied *(Gaunt, Stirling; Snigirev; Ceccopieri; Chang, Manohar, Waalewijn; Rinaldi, Scopetta,Vento; Golec-Biernat, Lewandowska)*.

The experimental effective cross section, $\sigma_{\rm eff}^{\rm exp}$, which is not measured directly but is extracted by means of the normalization to the product of two single cross sections:

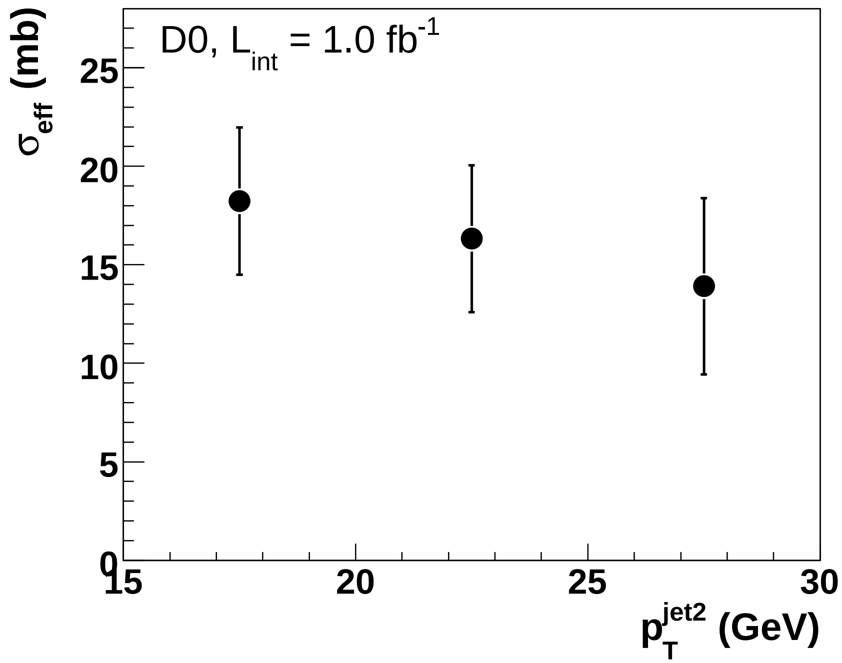
$$\frac{\boldsymbol{\sigma}_{DPS}^{\gamma+3j}}{\boldsymbol{\sigma}^{\gamma j}\boldsymbol{\sigma}^{jj}} = [\boldsymbol{\sigma}_{\rm eff}^{\rm exp}]^{-1},$$

appears to be dependent on the probing hard scale. It should DECREASE with increasing the resolution scale because all additional contributions to the cross section of double parton scattering are positive and increase.

In the above formula, $\sigma^{\gamma j}$ and σ^{jj} are the inclusive $\gamma+$ jet and dijets cross sections, $\sigma_{DPS}^{\gamma+3j}$ is the inclusive cross section of the $\gamma{+}3$ jets events produced in the double parton process.

It is worth noticing that the CDF and D0 Collaborations extract $\sigma_{\rm eff}^{\rm exp}$ without any theoretical predictions on the $\gamma+$ jet and dijets cross sections, by comparing the number of observed double parton $\gamma + 3$ jets events in ONE $p\bar{p}$ collision to the number of $\gamma+$ jet and dijets events occurring in TWO separate $p\bar{p}$ collisions.

The recent D0 measurements represent this effective cross section, $\sigma_{\rm eff}^{\rm exp}$, as a function of the second (ordered in the transverse momentum, p_T) jet p_T, $p_T^{\rm jet2}$, which can serve as a resolution scale. The obtained cross sections reveal a tendency to be dependent on this scale.



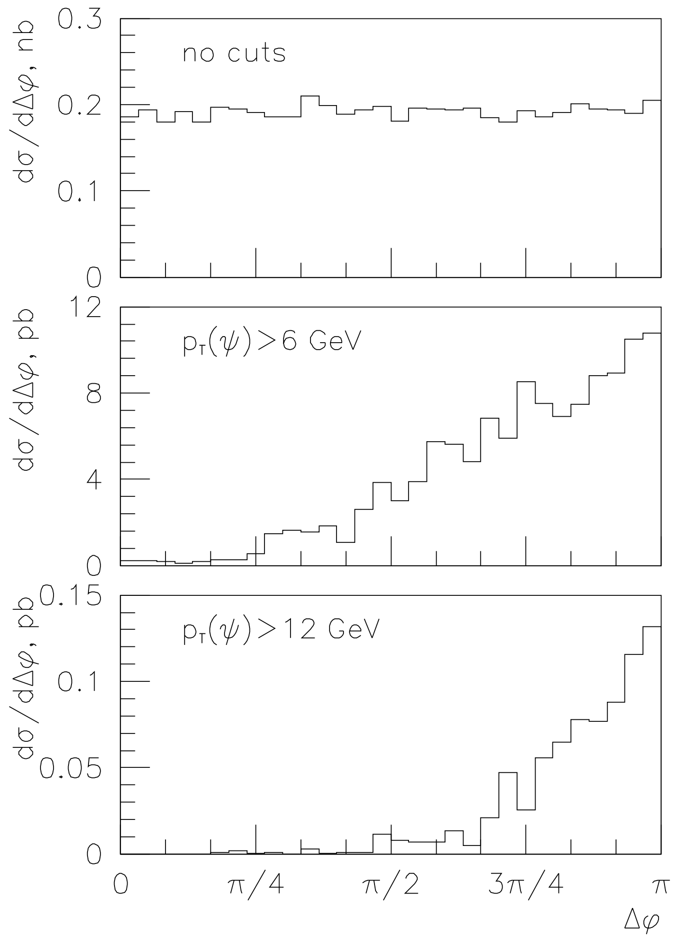
This observation can be interpreted as the first indication to the QCD evolution of double parton distributions
(*Snigirev; Flensburg, Gustafson, Lonnblad, Ster*).

Promising candidate processes to probe DPS at the LHC:

- same-sign W production ("pure", BUT very rare)
- γ + 3 jets (Tevatron also: D0, CDF)
- $W(Z)$ + 2 jets (ATLAS — first measurement σ_{eff} at LHC)
- 4 jets (Tevatron also: CDF)
- $b\bar{b}$ pair +2 jets
- $b\bar{b}$ pair + W boson
- pairs of heavy mesons (in particular, double J/ψ production) (LHCb — first measurement of double J/ψ production)

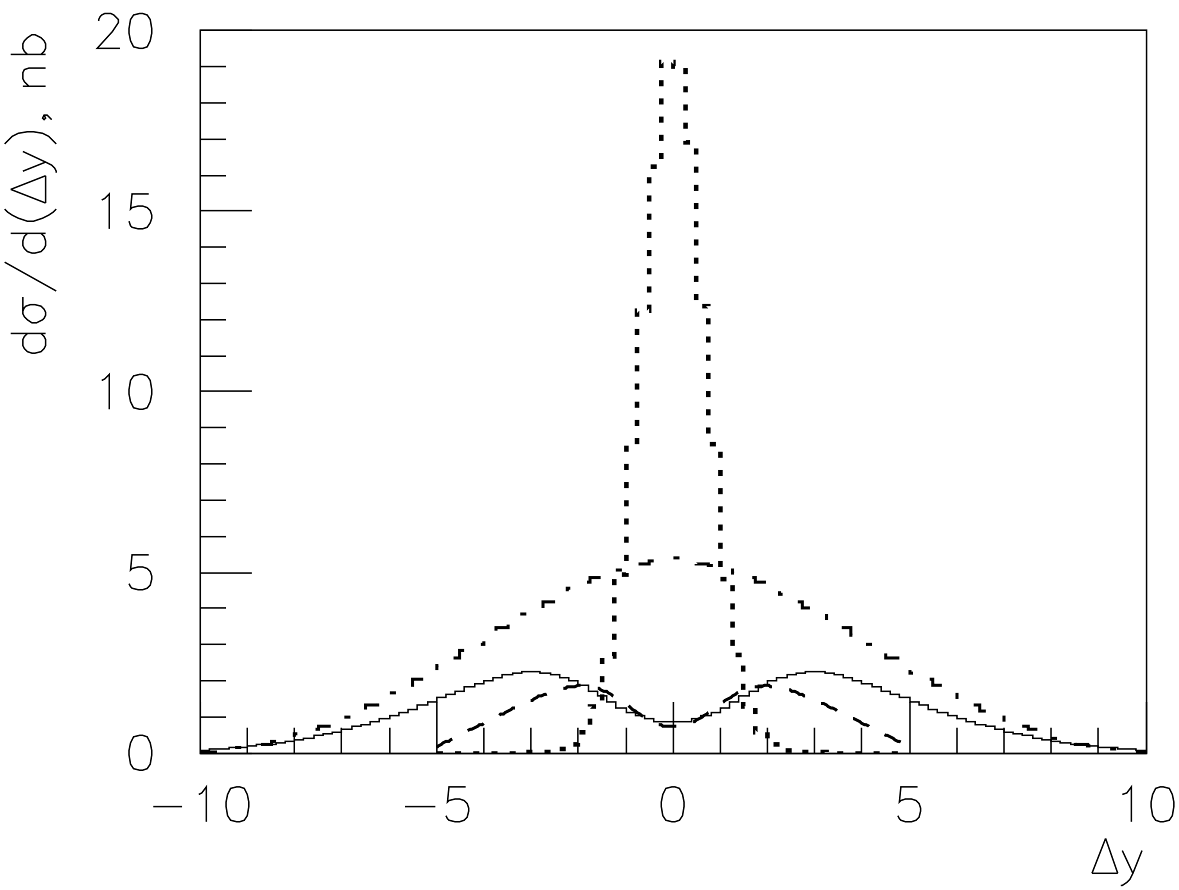
J/ψ pairs production

Azimuthal angle difference distribution after imposing cuts on the J/ψ transverse momenta for SPS



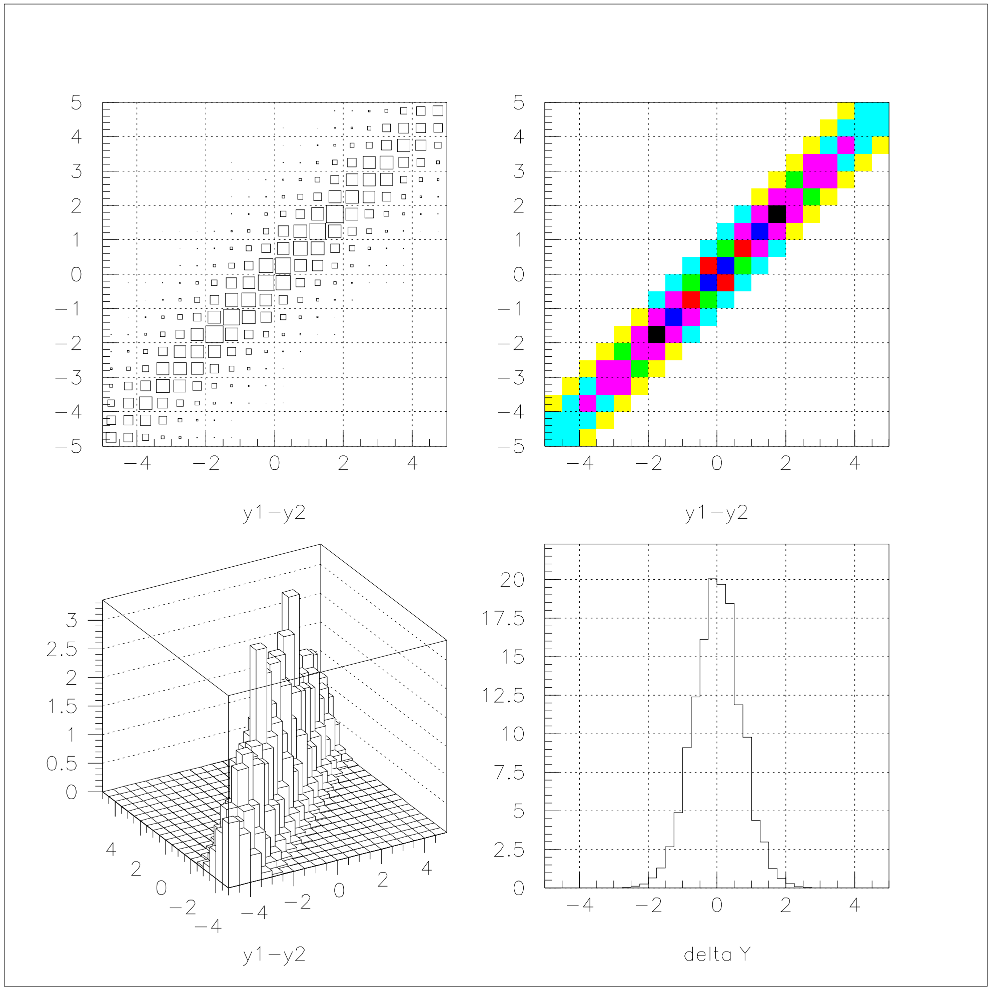
It is rather difficult to disentangle the SPS and DPS (flat) modes: the difference becomes visible only at sufficiently high cuts, where the production rates are, indeed, very small.

Distribution over the rapidity difference between J/ψ mesons. (Dotted curve: leading-order SPS, dash-dotted curve: DPS)

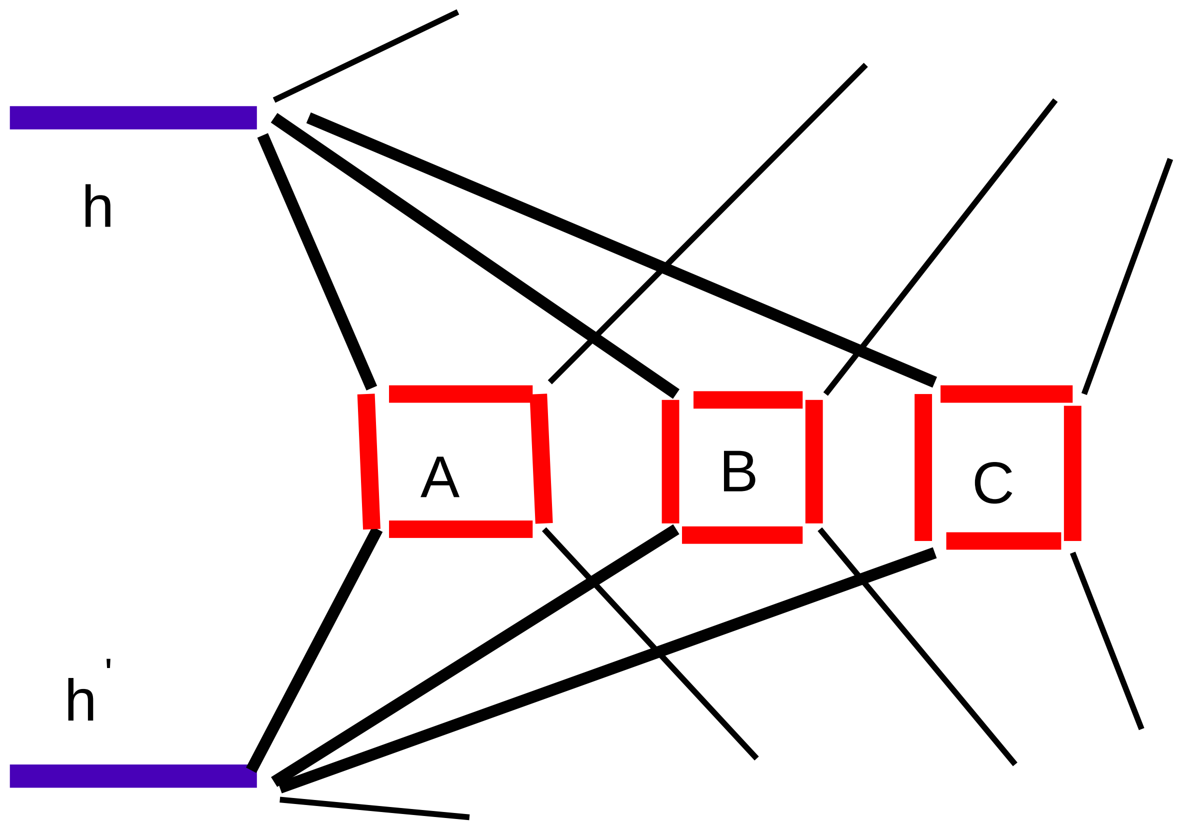


Selecting large rapidity difference events looks more promising to disentangle the SPS and DPS modes

Double differential distribution for the leading-order SPS production mode



It is qiute possible (BUT very rarely): hard TRIPLE parton scattering (subprocesses A, B and C)



Similar to DPS with only the assumption of factorization of the three hard parton subprocesses A, B and C, the inclusive cross section of a TPS process in a hadron collision may be written in the following form

$$\boldsymbol{\sigma}_{\mathrm{TPS}}^{(A,B,C)} = \sum_{i,j,k,l,m,n} \int \Gamma_{ijk}(x_1, x_2, x_3; \mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3; Q_1^2, Q_2^2, Q_3^2)$$

$$\times \hat{\sigma}_{il}^{A}(x_1, x_1', Q_1^2)\hat{\sigma}_{jm}^{B}(x_2, x_2', Q_2^2)\hat{\sigma}_{kn}^{C}(x_3, x_3', Q_3^2)$$

$$\times \Gamma_{lmn}(x_1', x_2', x_3'; \mathrm{b}_1 - \mathrm{b}, \mathrm{b}_2 - \mathrm{b}, \mathrm{b}_3 - \mathrm{b}; Q_1^2, Q_2^2, Q_3^2)$$

$$\times dx_1 dx_2 dx_3 dx_1' dx_2' dx_3' d^2b_1 d^2b_2 d^2b_3 d^2b.$$

Here $\Gamma_{ijk}(x_1, x_2, x_3; \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}; Q_1^2, Q_2^2, Q_3^2)$ are the triple parton distribution functions, which depend on the longitudinal momentum fractions x_1, x_2 and x_3 and on the transverse position $\mathbf{b_1}$, $\mathbf{b_2}$ and $\mathbf{b_3}$ of the three partons i, j and k undergoing the hard subprocesses A, B and C at the scales Q_1, Q_2 and Q_3. $\hat{\sigma}_{il}^{A}$, $\hat{\sigma}_{jm}^{B}$ and $\hat{\sigma}_{kn}^{C}$ are the parton-level subprocess cross sections.**b** is the impact parameter — the distance between centres of colliding (e.g., the beam and the target) hadrons in transverse plane.
The appropriate combinatorial factor $(m/3!)$ should be taken into account in the case of the indistinguishable final states.

As in the case of DPS it is typically taken that the triple parton distribution functions may be decomposed in terms of the longitudinal and transverse components as follows:

$$\Gamma_{ijk}(x_1, x_2, x_3; \mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3; Q_1^2, Q_2^2, Q_3^2)$$

$$= D_h^{ijk}(x_1, x_2, x_3; Q_1^2, Q_2^2, Q_3^2) f(\mathrm{b}_1) f(\mathrm{b}_2) f(\mathrm{b}_3),$$

where $f(\mathbf{b}_1)$ is supposed to be an universal function for all kind of partons as before.

If one makes the further assumption that the longitudinal components $D_h^{ijk}(x_1, x_2, x_3; Q_1^2, Q_2^2, Q_3^2)$ reduce to the product of three independent single parton distributions,

$$D_h^{ijk}(x_1, x_2, x_3; Q_1^2, Q_2^2, Q_3^2) = D_h^i(x_1; Q_1^2) D_h^j(x_2; Q_2^2) D_h^k(x_3; Q_3^2)$$

the cross section of TPS can be expressed in the simple form

$$\sigma_{\mathrm{TPS}}^{(A,B,C)} = \frac{\sigma_{\mathrm{SPS}}^{A} \sigma_{\mathrm{SPS}}^{B} \sigma_{\mathrm{SPS}}^{C}}{\sigma_{\mathrm{TPS,fact}}^{2}}$$

$$\boldsymbol{\sigma}^2_{\mathrm{TPS,fact}} = [\int d^2b (T(\mathrm{b}))^3]^{-1}.$$

$$\boldsymbol{\sigma}_{\mathrm{eff}} = [\int d^2b (T(\mathrm{b}))^2]^{-1}$$

$$\boldsymbol{\sigma}_{\mathrm{TPS,fact}} = k \cdot \boldsymbol{\sigma}_{\mathrm{eff}}$$

with $k = 0.82 \pm 0.11$ as the average of all typical parton transverse profiles usually used in the literature (Gaussian, dipole fit, PYTHIA, HERWIG,....)

TPS in QCD:

A.M. Snigirev, Phys. Rev. D 94, 034026 (2016).

D. d'Enterria, A.M. Snigirev, arXiv:1612.05582 [hep-ph] (2016) (PRL 118, 122001 (2017)).

D. d'Enterria, A.M. Snigirev, arXiv:1612.08112 [hep-ph] (2016).

m-parton distributions:

$$\frac{dD_i^{j_1 \ldots j_m}(x_1, \ldots, x_m, t)}{dt} = \sum_{l=1}^{m} \sum_{j'} \int_{x_l}^{1-x_1-\ldots-x_{l-1}-x_{l+1}-\ldots-x_m} \frac{dx'}{x'} \times$$

$$\times D_i^{j_1 \ldots j_{l-1} j' j_{l+1} \ldots j_m}(x_1, \ldots, x_{l-1}, x', x_{l+1}, \ldots, x_m, t) P_{j' \to j_l}\left(\frac{x_l}{x'}\right)$$

$$+ \sum_{l=1}^{m} \sum_{p=l+1}^{m} \sum_{j'} \frac{1}{x_l + x_p} P_{j' \to j_l j_p}\left(\frac{x_l}{x_l + x_p}\right) \times$$

$$\times D_i^{j_1 \ldots j_{l-1} j' j_{l+1} \ldots j_{p-1} j_{p+1} \ldots j_m}(x_1, \ldots, x_{l-1}, x_l + x_p, x_{l+1}, \ldots, x_{p-1}, x_{p+1}, \ldots, x_m, t)$$

Shelest, Snigirev, Zinovjev, Preprint ITP-83-46E, Kiev, 1983

allows us:

1) to justify the simple factorization form of cross sections
2) to write the evolution corrections to it

BACK UP

$$\sigma^{(A,B,C)}_{\mathrm{TPS,fact.00}} = \sum_{i,j,k,l,m,n} \int D^i_h(x_1, Q^2_0, Q^2_1) D^j_h(x_2, Q^2_0, Q^2_2)$$

$$\times D^k_h(x_3, Q^2_0, Q^2_3)(2\pi)^2 \delta(\mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3) \hat{\sigma}^A_{il}(x_1, x^{'}_1, Q^2_1)$$

$$\times \hat{\sigma}^B_{jm}(x_2, x^{'}_2, Q^2_2) \hat{\sigma}^C_{kn}(x_3, x^{'}_3, Q^2_3)$$

$$\times D^l_{h'}(x^{'}_1, Q^2_0, Q^2_1) D^m_{h'}(x^{'}_2, Q^2_0, Q^2_2) D^n_{h'}(x^{'}_3, Q^2_0, Q^2_3)$$

$$\times F_{2g}(\mathrm{q}_1) F_{2g}(\mathrm{q}_2) F_{2g}(\mathrm{q}_3) F_{2g}(-\mathrm{q}_1) F_{2g}(-\mathrm{q}_2) F_{2g}(-\mathrm{q}_3)$$

$$\times dx_1 dx_2 dx_3 dx^{'}_1 dx^{'}_2 dx^{'}_3 \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{d^2 q_3}{(2\pi)^2}.$$

Making the Fourier transformation

$$f(\mathrm{b_i}) = \int e^{-i\mathrm{b_i}\cdot\mathrm{q_i}} F_{2g}(\mathrm{q_i}) \frac{d^2 q_i}{(2\pi)^2}, i = 1, 2, 3,$$

one derives immediately the factorization result

$$\sigma^{(A,B,C)}_{\mathrm{TPS},10+01}$$

$$= \sum_{i,j,k,l,m,n} \int \sum_{j' j_1' j_2'} \int_{Q_0^2}^{\min(Q_1^2,Q_2^2)} \frac{\alpha_s(k^2)}{2\pi k^2} dk^2 \int_{x_1}^{1} \frac{dz_1}{z_1} \int_{x_2}^{1} \frac{dz_2}{z_2} \theta(1 - z_1 - z_2)$$

$$\times D_h^{j'}(z_1 + z_2, Q_0^2, k^2) \frac{1}{z_1 + z_2} P_{j' \to j_1' j_2'}\left(\frac{z_1}{z_1 + z_2}\right)$$

$$\times D^i_{j_1'}\left(\frac{x_1}{z_1}, k^2, Q_1^2\right) D^j_{j_2'}\left(\frac{x_2}{z_2}, k^2, Q_2^2\right) D^k_h(x_3, Q_0^2, Q_3^2)$$

$$\times \hat{\sigma}^A_{il}(x_1, x_1', Q_1^2) \hat{\sigma}^B_{jm}(x_2, x_2', Q_2^2) \hat{\sigma}^C_{kn}(x_3, x_3', Q_3^2)$$

$$\times D^l_{h'}(x_1', Q_0^2, Q_1^2) D^m_{h'}(x_2', Q_0^2, Q_2^2) D^n_{h'}(x_3', Q_0^2, Q_3^2) (2\pi)^2 \delta(\mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3)$$

$$\times F_{2g}(\mathrm{q}_1 + \mathrm{q}_2) F_{2g}(\mathrm{q}_3) F_{2g}(-\mathrm{q}_1) F_{2g}(-\mathrm{q}_2) F_{2g}(-\mathrm{q}_3)$$

$$\times dx_1 dx_2 dx_3 dx_1' dx_2' dx_3' \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{d^2 q_3}{(2\pi)^2} + \ldots$$

where "+..." denotes another 5 analogous contributions

The scale factor for single splitting contributions

$$\frac{1}{\sigma^2_{\mathrm{TPS,10}}} = \int (2\pi)^2 \delta(\mathrm{q}_1 + \mathrm{q}_2 + \mathrm{q}_3) F_{2g}(\mathrm{q}_1 + \mathrm{q}_2) F_{2g}(\mathrm{q}_3)$$

$$\times F_{2g}(-\mathrm{q}_1) F_{2g}(-\mathrm{q}_2) F_{2g}(-\mathrm{q}_3) \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{d^2 q_3}{(2\pi)^2}$$

$$= \int f^2(\mathrm{b} - \mathrm{b}_3 + \mathrm{b'}_3) f(\mathrm{b}) f(\mathrm{b}_3) f(\mathrm{b'}_3) d^2b d^2b_3 d^2b'_3.$$

In a simple model where the transverse parton density is taken to have Gaussian functional form, the ratio

$$\frac{\sigma^2_{\mathrm{TPS,fact.00}}}{\sigma^2_{\mathrm{TPS,10}}} = \frac{12}{7} = 1.7$$

shows that the single splitting contributions to the cross section are enhanced, relative to the factorization one, by the factor

$$2 \times 3(\text{combinatorial}) \times 1.7(\text{scale}) \sim 10.$$

So we conclude that the single splitting terms may provide a sizable contribution to the cross section of TPS even if they constitute a small correction to the triple factorized parton distribution functions.