Weyl quantization map and star product for the charge-monopole system

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M.A. Soloviev

P.N. Lebedev Institute, Moscow

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Motivation and historical remarks

Dirac's charge quantization condition:

$$eg = \frac{1}{2}n\hbar c$$

- The fibre bundle description (**Wu & Yang (1975), Greub & Petry (1975), Trautman (1977),...**)
- The magnetic Weyl calculus ($\mathbf{B} = \nabla \times \mathbf{A}$) (**Stratonovich, Müller (1999), Karasev & Osborn (2002), Măntoiu & Purice (2004), ...**)
- The quaternionic Hilbert space formulation (**Emch & Jadczyk (1998), Cariñena, Gracia-Bondia, Lizzi, Marmo, and Vitale (2009)...**)
- The theory of deformation quantization of Poisson manifolds (**Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer (1978), Fedosov, Rieffel (1993), Kontsevich (1997), ...**)

The aims are: **1)** to define rigorously Weyl quantization maps by using complex and quaternionic Hilbert spaces; **2)** to derive representations for the corresponding star product; **3)** to show how this product is related to the Kontsevich deformation quantization formula

Magnetic Poisson brackets

Hamilton's equations of motion for a particle in a magnetic field

$$\dot{x}^i = \{x^i, H\}, \qquad \dot{p}_i = \{p_i, H\}$$

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2)$$

The magnetic symplectic form

$$dp_i \wedge dx^i + \frac{1}{2}\beta_{ij}\, dx^i \wedge dx^j, \quad \beta_{ij} = e\,\epsilon_{ijk} B^k$$

$$\Downarrow$$

$$\{x^i, x^j\} = 0, \quad \{p_i, p_j\} = \beta_{ij}(x), \quad \{x^i, p_j\} = \delta^i_j$$

The magnetic monopole field

$$B^k(x) = g\,\frac{x^k}{|x|^3}$$

$$\mathbf{A}_+(r,\phi,\theta) = \frac{g}{r}\tan\frac{\theta}{2}\,\mathbf{e}_\phi \ (\theta \neq \pi), \quad \mathbf{A}_-(r,\phi,\theta) = -\frac{g}{r}\cot\frac{\theta}{2}\,\mathbf{e}_\phi \ (\theta \neq 0)$$

$$\mathbf{A}_+ = \mathbf{A}_- + 2g\,\mathrm{grad}\,\phi \quad (\theta \neq 0, \pi)$$

Description in terms of fiber bundle theory

$$\Psi_+ = e^{in\phi}\Psi_-, \quad n = 2eg/\hbar$$
$$P_j = -i\hbar\nabla_j, \qquad \nabla_j = \partial_j - i\frac{e}{\hbar}\mathbf{A}_j$$
$$[Q^i, Q^j] = 0, \quad [Q^i, P_j] = i\hbar\,\delta^i_j, \quad [P_i, P_j] = i\hbar\,\beta_{ij},$$

The underlying principal bundle is $\dot{\mathbb{C}}^2 \overset{\text{def}}{=} \mathbb{C}^2 \setminus \{0\}$ **equipped with the projection** π **onto the base space** $\dot{\mathbb{R}}^3 \overset{\text{def}}{=} \mathbb{R}^3 \setminus \{0\} = \dot{\mathbb{C}}^2/U(1)$

$$\dot{\mathbb{C}}^2 \xrightarrow{\pi} \dot{\mathbb{R}}^3 : \quad x^j = z^\dagger \sigma_j z$$

The restriction of the bundle $\left(\dot{\mathbb{C}}^2, \dot{\mathbb{R}}^3, \pi, U(1)\right)$ to the unit sphere is the **Hopf bundle**

$$S^3 \approx SU(2) \longrightarrow SU(2)/U(1) \approx S^2$$

The natural connection

$$\omega = i\,\mathrm{Im}(z^\dagger dz)/z^\dagger z$$

Local cross-sections

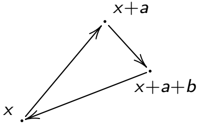
$$s_+ : \; z_1 = \sqrt{r}\cos(\theta/2), \; z_2 = \sqrt{r}\sin(\theta/2)e^{i\phi}, \qquad s_- = s_+ e^{i\phi}$$
$$n\, s^*_{(\pm)}\omega = i\frac{e}{\hbar}\mathbf{A}^{(\pm)}_j dx^j$$

A weak projective representation of the translation group

Definition. **For any vector $a \in \mathbb{R}^3$, we define $V(a)$ to be the operator transforming each section Ψ of the line bundle into another section whose value at the point x is the parallel transport of the value of Ψ at the point $x + a$ along the straight line path $x_t = x + a - ta$, $0 \leq t \leq 1$.**

$$\Big(V(a)V(b)V(a+b)^{-1}\Psi\Big)(x) = \exp\left\{-\frac{ie}{\hbar}\oint_{\partial\triangle(x;a,b)} \mathbf{A}_\pm \cdot d\mathbf{r}\right\}\Psi(x)$$

Triangle $\triangle(x; a, b)$ in the base x-space



$$\exp\left\{-\frac{ie}{\hbar}\oint_{\partial\triangle(x;a,b)} \mathbf{A}_\pm \cdot d\mathbf{r}\right\} = \exp\left\{-\frac{ie}{\hbar}\int_{\triangle(x;a,b)} \mathbf{B} \cdot d\mathbf{n}\right\}$$

$$V(a)V(b) = M(a,b)V(a+b), \quad \big(M(a,b)\Psi\big)(x) = \exp\left\{-\frac{i}{\hbar}\int_{\triangle(x;a,b)} \beta\right\}\Psi(x)$$

Associativity and charge quantization

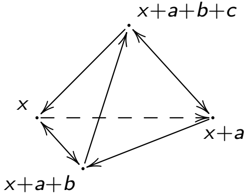
The associativity of the operator product $(V(a)V(b))V(c) = V(a)(V(b)V(c))$ implies that the multiplier $M(a,b)$ satisfies the **2-cocycle relation**

$$M(a,b)M(a+b,c) = V(a)M(b,c)V(a)^{-1}M(a,b+c),$$

where $V(a)M(b,c)V(a)^{-1}$ is the operator of multiplication by

$$\exp\left\{-\frac{i}{\hbar}\int_{\triangle(x+a;b,c)} \beta\right\}$$

The cocycle identity means that the flux through the surface of the tetrahedron spanned by the points x, $x+a$, $x+a+b$, and $x+a+b+c$ is an integer multiple of 2π, which is satisfied by the charge quantization condition



The functional analytic aspects

1. The set of smooth compactly supported sections of the line bundle E_n associated with the principal bundle $(\dot{\mathbb{C}}^3, \dot{\mathbb{R}}^3, \pi, U(1))$ is dense in the space $L^2(\dot{\mathbb{R}}^3, E_n)$ of square integrable sections.

2. The space $L^2(\dot{\mathbb{R}}^3, E_n)$ is naturally isomorphic to the Hilbert space of complex-valued functions on $\dot{\mathbb{C}}^2$ satisfying the equivariance condition

$$\Psi(ze^{i\alpha}) = e^{in\alpha}\Psi(z), \qquad z \in \dot{\mathbb{C}}^2,$$

and square integrable with the weight $z^\dagger z/\pi$.

3. For each fixed a, the operator-valued function $V(ta)$, $t \in \mathbb{R}$, is a strongly continuous one-parameter unitary group, i.e.,

$$V(sa)V(ta) = V((s+t)a) \text{ for all } s, t \in \mathbb{R} \text{ and } V(ta)\Psi \to \Psi \text{ as } t \to 0$$

4. The map $t \to V(ta)$ is strongly differentiable at $t = 0$ on the set of smooth sections with compact support and its derivative is ∇_a.

5. The operator $i\nabla_a$ is essentially self-adjoint on the domain D consisting of all infinitely differentiable sections whose support does not intersect the line spanned by the vector a and its closure is the infinitesimal generator of the unitary group $V(ta)$.

Weyl quantization map

The magnetic Weyl system

$$T(u,v) = V(\hbar u)e^{iv\cdot Q}e^{-i\hbar u\cdot v/2} = e^{i(u\cdot P + v\cdot Q)}, \quad P = -i\hbar\nabla$$

forms a weak projective representation of the phase-space translations

$$T(w)T(w') = \mathcal{M}_\hbar(Q;w,w')T(w+w'), \qquad w=(u,v)$$

with the operator-valued multiplier

$$\Big(\mathcal{M}_\hbar(Q;w,w')\Psi\Big)(x) = \exp\left\{\frac{i\hbar}{2}(u\cdot v' - v\cdot u') - \frac{i}{\hbar}\int_{\triangle(x;\hbar u,\hbar u')}\beta\right\}\Psi(x)$$

The quantization map is defined by

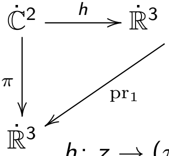
$$f \longmapsto \mathcal{O}_f = \frac{1}{(2\pi)^3}\int dudv\, \tilde{f}(u,v)\, e^{i(u\cdot P + v\cdot Q)},$$

where

$$\tilde{f}(u,v) = \frac{1}{(2\pi)^3}\int dpdq\, e^{-i(p\cdot u + q\cdot v)} f(p,q)$$

Quaternionic representation

The principal bundle $(\dot{\mathbb{C}}^2, \dot{\mathbb{R}}^3, \pi, U(1))$ is a reduction of the trivial bundle $\dot{\mathbb{R}}^3 \times SU(2)$



$$h(z \cdot e^{i\alpha}) = h(z) \cdot \eta(e^{i\alpha}), \quad \eta \colon U(1) \to SU(2)$$

$$h \colon z \to (\pi(z), g), \qquad \text{where} \quad g = \frac{1}{|z|}\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \in SU(2)$$

$$(h^*\Omega)(\xi) = \eta_*(\omega(\xi)), \qquad \eta_* \colon \operatorname{Im}\mathbb{C} \to \mathfrak{su}(2), \quad \eta_*(i) = i\sigma_3$$

$$s_+^*\omega = \frac{i}{2}(1 - \cos\theta)d\phi \quad \Rightarrow \quad (h \circ s_+)^*\Omega = \frac{i}{2}\sigma_3(1 - \cos\theta)d\phi$$

Transforming the local cross-section $h \circ s_+$ into the global one

$$s = (h \circ s_+)g, \quad g(\theta, \phi) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\phi} \\ -\sin(\theta/2)e^{i\phi} & \cos(\theta/2) \end{pmatrix},$$

we obtain the $\mathfrak{su}(2)$-potential

$$g^{-1}(h \circ s_+)^*\Omega\, g + g^{-1}dg = -\frac{i}{2}\epsilon_{ijk}\frac{x^i}{|x|^2}\sigma^j dx^k$$

Quaternionic quantization map

Quaternionic imaginary units $\quad \mathbf{e}_j = -i\sigma_j, \quad j = 1, 2, 3$

The connection Ω induces on $L^2(\dot{\mathbb{R}}^3, d^3x; \mathbb{H})$ the covariant derivative

$$\boldsymbol{\nabla}_k = \partial_k + \frac{1}{2}\epsilon_{ijk}\frac{x^i}{|x|^2}$$

whose components satisfy the commutation relations

$$[\boldsymbol{\nabla}_i, \boldsymbol{\nabla}_j] = -\frac{1}{2}\mathbf{J}(x)\epsilon_{ijk}\frac{x^k}{|x|^3}, \qquad \mathbf{J}(x) = \frac{x^k\mathbf{e}_k}{|x|} \quad (\mathbf{J}^2 = -1)$$

The operators $\mathrm{V}(a) = \exp\{\boldsymbol{\nabla}_a\}$ form a weak quaternionic projective representation of the translation group

$$\mathrm{V}(a)\mathrm{V}(b) = \mathbf{M}(a,b)\mathrm{V}(a{+}b), \quad \big(\mathbf{M}(a,b)\Psi\big)(x) = \exp\left\{-\frac{\mathbf{J}(x)}{\hbar}\int_{\triangle(x;a,b)}\beta\right\}\Psi(x)$$

The quaternionic quantization map is defined by

$$f \longmapsto \mathbf{O}_f = \frac{1}{(2\pi)^3}\int dudv\,\tilde{\mathbf{f}}(u,v)\,e^{\mathbf{J}(x)(u\cdot\mathbf{P}+v\cdot\mathbf{Q})}, \qquad \mathbf{P} = -\mathbf{J}(x)\hbar\,\boldsymbol{\nabla},$$

where

$$\tilde{\mathbf{f}}(u,v) = \frac{1}{(2\pi)^3}\int dpdq\,e^{-\mathbf{J}(x)(p\cdot u+q\cdot v)}\big(\operatorname{Re} f(p,q) + \mathbf{J}(x)\operatorname{Im} f(p,q)\big).$$

An operator analog of the twisted convolution

The product of the operators corresponding to the phase-space functions f and g can be written as

$$\mathcal{O}_f \mathcal{O}_g = \frac{1}{(2\pi)^6} \int dw dw' \, \tilde{f}(w) \tilde{g}(w') T(w) T(w')$$

$$= \frac{1}{(2\pi)^6} \int dw dw' \, \tilde{f}(w) \tilde{g}(w') \mathcal{M}_\hbar(Q; w, w') T(w + w')$$

$$= \frac{1}{(2\pi)^3} \int dw \, (\tilde{f} \circledast_\hbar \tilde{g})(Q; w) T(w),$$

where $w = (u, v), \; T(w) = e^{i(u\cdot P + v\cdot Q)}$, **and**

$$(\tilde{f} \circledast_\hbar \tilde{g})(Q; w) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^3} \int dw' \, \tilde{f}(w - w') \tilde{g}(w') \mathcal{M}_\hbar(Q; w - w', w').$$

Lemma. **Let $\mu(Q)$ be the multiplication operator by a complex-valued function $\mu(x)$. Then the symbol of the product $\mu(Q)T(u, v)$ is equal to the function**

$$\mu(x - \hbar u/2) e^{i(u\cdot p + v\cdot x)}$$

The integral representation of the star product

$$(f\star_\hbar g)(s) = \frac{1}{(2\pi)^6}\int dw dw'\, \tilde{f}(w-w')\tilde{g}(w')\mathcal{M}_\hbar(x-\hbar u/2; w-w', w')e^{iw\cdot s}$$

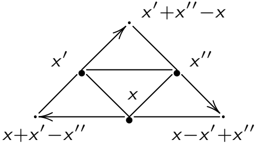
$$= \frac{1}{(2\pi)^6}\int dw dw'\, \tilde{f}(w)\tilde{g}(w')\mathcal{M}_\hbar(x-\hbar(u+u')/2, w, w')e^{i(w+w')\cdot s} \qquad (s=(x,p)).$$

$$\Downarrow$$

$$(f\star_\hbar g)(s) = \int ds'\, ds''\, K(s;s',s'')\, f(s')g(s''),$$

$$K(x,p;x',p',x'',p'') = \frac{1}{(\pi\hbar)^6}\exp\left\{\frac{2i}{\hbar}[(x-x')(p-p'')-(x-x'')(p-p')]\right\}$$
$$\times\exp\left\{-\frac{i}{\hbar}\int_{\widehat{\triangle}(x,x',x'')}\beta\right\}$$

Triangle $\widehat{\triangle}(x,x',x'')$ in the base space



The asymptotic expansion of the star product

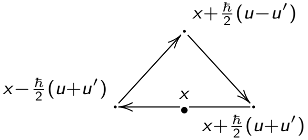
The differential form of the star product is obtained from the shifted multiplier

$$\mathcal{M}_\hbar(x-\hbar(u+u')/2; u, v, u', v') = \exp\left\{ \frac{i\hbar}{2}(u\cdot v' - v\cdot u') - \frac{i}{\hbar}\int_{\triangle^*(x;\hbar u,\hbar u')} \beta \right\}$$

by substituting

$$u \to -i\overleftarrow{\partial_p}, \quad v \to -i\overleftarrow{\partial_x}, \quad u' \to -i\overrightarrow{\partial_p}, \quad v' \to -i\overrightarrow{\partial_x}$$

Triangle $\triangle^*(x;\hbar u,\hbar u')$



The power expansion of the magnetic flux through the triangle gives the same result as the Zassenhaus formula

$$e^{i(u\cdot P+u'\cdot P)} = e^{iu\cdot P}e^{iu'\cdot P}\prod_{n=2}^{\infty} e^{C_n(u\cdot P,u'\cdot P)},$$

$$C_2 = \frac{1}{2}[uP, u'P] = \frac{i\hbar}{2}u^i\beta_{ij}u'^j = \frac{i\hbar}{2}u\cdot\beta u',$$

$$C_3 = -\frac{i}{6}[uP,[uP,u'P]] - \frac{i}{3}[u'P,[uP,u'P]] = -\frac{i\hbar^2}{6}\Big(u\cdot(u\cdot\partial)\beta u' + 2u\cdot(u'\cdot\partial)\beta u'\Big)$$

Calculation up to the $\hbar^3$-order terms

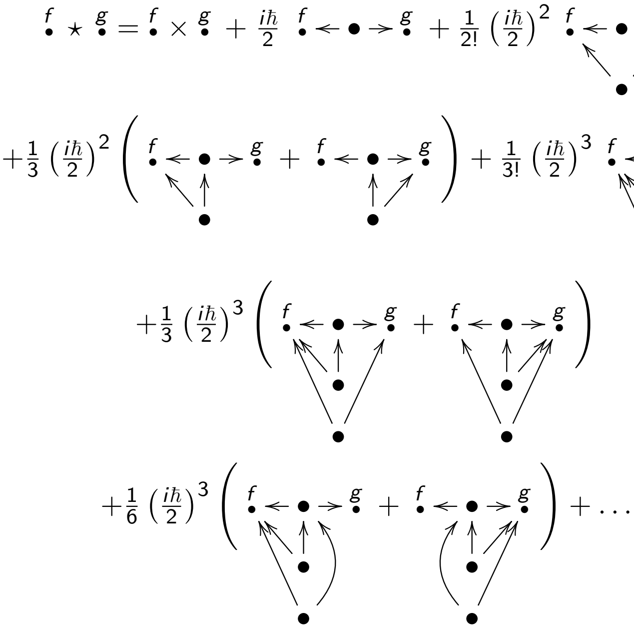
The bidifferential operator defining the third order star product

$$\sum_{k=0}^{3} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\cdot\overrightarrow{\partial_x} + \overleftarrow{\partial_p}\cdot\beta\overrightarrow{\partial_p}\right)^k$$
$$+\frac{\hbar^2}{12}\left(\overleftarrow{\partial_p}\cdot(\overleftarrow{\partial_p}\cdot\partial)\beta\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\cdot(\overrightarrow{\partial_p}\cdot\partial)\beta\overrightarrow{\partial_p}\right)\left(1+\frac{i\hbar}{2}(\overleftarrow{\partial_x}\cdot\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\cdot\overrightarrow{\partial_x} + \overleftarrow{\partial_p}\cdot\beta\overrightarrow{\partial_p})\right)$$
$$-\frac{i\hbar^3}{48}\left(\overleftarrow{\partial_p}\cdot(\overleftarrow{\partial_p}\cdot\partial)^2\beta\overrightarrow{\partial_p} + \overleftarrow{\partial_p}\cdot(\overrightarrow{\partial_p}\cdot\partial)^2\beta\overrightarrow{\partial_p}\right)$$

The explicit expression in terms of the Poisson tensor $\mathcal{P} = \begin{pmatrix} \beta(x) & -I \\ I & 0 \end{pmatrix}$

$$f\star g = fg + \frac{i\hbar}{2}\mathcal{P}^{ab}\partial_a f\partial_b g - \frac{\hbar^2}{8}\mathcal{P}^{a_1b_1}\mathcal{P}^{a_2b_2}\partial_{a_1}\partial_{a_2}f\partial_{b_1}\partial_{b_2}g$$
$$-\frac{i\hbar^3}{48}\mathcal{P}^{a_1b_1}\mathcal{P}^{a_2b_2}\mathcal{P}^{a_3b_3}\partial_{a_1}\partial_{a_2}\partial_{a_3}f\partial_{b_1}\partial_{b_2}\partial_{b_3}g$$
$$-\frac{\hbar^2}{12}\mathcal{P}^{a_1b_1}\partial_{b_1}\mathcal{P}^{a_2b_2}\left(\partial_{a_1}\partial_{a_2}f\partial_{b_2}g - \partial_{a_2}f\partial_{a_1}\partial_{b_2}g\right)$$
$$-\frac{i\hbar^3}{24}\mathcal{P}^{a_1b_1}\mathcal{P}^{a_2b_2}\partial_{b_2}\mathcal{P}^{a_3b_3}\left(\partial_{a_1}\partial_{a_2}\partial_{a_3}f\partial_{b_1}\partial_{b_3}g - \partial_{a_1}\partial_{a_3}f\partial_{b_1}\partial_{a_2}\partial_{b_3}g\right)$$
$$-\frac{i\hbar^3}{48}\mathcal{P}^{a_1b_1}\mathcal{P}^{a_2b_2}\partial_{b_1}\partial_{b_2}\mathcal{P}^{a_3b_3}\left(\partial_{a_1}\partial_{a_2}\partial_{a_3}f\partial_{b_3}g + \partial_{a_3}f\partial_{a_1}\partial_{a_2}\partial_{b_3}g\right) + O(\hbar^4)$$

Kontsevich's graphical representation



Conclusions

- The Weyl quantization map can be rigorously defined for the charge-monopole system by using the parallel transport of fibers, which applies to the operator representations in both complex and quaternionic Hilbert spaces.

- These two operator quantizations yield the same phase-space star product whose integral form provides the strict deformation quantization of the system.

- A simple and direct way of finding the star product is by treating the magnetic Weyl system as a weak projective representation of the translation group and using an operator analog of the twisted convolution product.

- The associativity of the magnetic star product is ensured by the charge quantization condition.

- The differential form of this star product agrees completely with the Kontsevich formula for deformation quantization of Poisson manifolds.