Quantum properties of Lifshitz theories

Christian Friedrich Steinwachs

in collaboration with:
A. Barvinsky, D. Blas, M. Herrero-Valea,
D. Nesterov, G. Pérez-Nadal, S. Sibiryakov

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Outline

1. Hořava gravity

2. Renormalizability of projectable Hořava gravity

3. One-loop RG flow of $D = 2 + 1$ projectable Hořava gravity

4. Heat kernel technique for anisotropic operators

5. Conclusions

Quantum theories of gravity

Quantum Einstein gravity: perturbatively non-renormalizable

['t Hooft & Veltman (1974), Goroff & Sagnotti (1986)]

Einstein gravity as effective field theory: fundamental theory (UV) not known
Renormalize only finite number of couplings → still predictive for $E \ll M$

[Wilson (1971), Weinberg (1979)]

Asymptotic safety: might provide a non-perturbative UV-complete scenario if there is a UV fixed point with finite dimensional UV critical surface

[Weinberg, Wetterich, Reuter, Percacci, Saueressig,...]

Higher derivative gravity: modification of the fundamental theory of gravity
Improved UV behaviour: perturbatively renormalizable but ghost problem

[Stelle (1977), Pais & Uhlenbeck (1950)]

Hořava gravity: Lorentz invariance broken in the UV but emergent in the IR!?
Allow for higher spatial derivatives but restrict to 2_{nd} order time derivatives

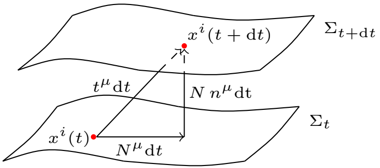
$$\mathcal{P} = \frac{1}{\omega^2 + k^2} - \frac{1}{\omega^2 + k^2} G k^{2z} \frac{1}{\omega^2 + k^2} + \ldots = \frac{1}{\omega^2 + k^2 + G k^{2z}}$$

Propagator "dressed" with higher spatial derivatives → no ghost problem

[Hořava (2009)]

ADM decomposition of GR

$$\mathrm{d}s^2 = N^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t), \quad i,j = 1,...,d$$



Gauss-Codazzi relation and extrinsic curvature K_{ij}

$$R(g) = K_{ij}K^{ij} - K^2 + R(\gamma) + \text{total derivative}$$

$$K_{ij} = \frac{1}{2\,N}\left(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \quad K := \gamma^{ij} K_{ij}$$

ADM decomposition of GR in $D = 3+1$-dimensions

$$S_{\mathrm{EH}} = \frac{M_{\mathrm{P}}^2}{2} \int \mathrm{d}t\, \mathrm{d}^3x\, N\, \gamma^{1/2} \Big(\underbrace{K_{ij}K^{ij} - K^2}_{\text{“kinetic” term}} + \underbrace{R(\gamma)}_{\text{“potential”}} \Big)$$

Hořava gravity

Fundamental Lorentz invariance broken: anisotropic scaling [Lifshitz (1941)]

$$t \to b^{-z}\, t, \qquad x^i \to b^{-1}\, x^i, \qquad i = 1, ..., d$$

Preferred time direction: invariance under foliation preserving diffeomorphisms

$$\text{FDiff}(\mathcal{M}): \; x^i \to \tilde{x}^i(\vec{x}, t), \quad t \to \tilde{t}(t)$$

Euclidean Hořava gravity in $D = d + 1$ ("projectable" version: $N(\vec{x}, t) = N(t)$)

$$S_{\mathrm{HG}} = \frac{1}{2\,G} \int \mathrm{d}t\mathrm{d}^d x\, \gamma^{1/2}\, N \Big\{ \underbrace{K_{ij}K^{ij} - \lambda\, K^2}_{\text{kinetic term}} + \underbrace{\mathcal{V}^{(d)}}_{\text{potential}} \Big\}$$

Structure of potential $\mathcal{V}^{(d)}$ depends on spatial dimension d

$$\begin{aligned} \mathcal{V}^{(d=2)} &= 2\Lambda + \mu R^2, \\ \mathcal{V}^{(d=3)} &= 2\Lambda - \eta R + \mu_1 R^2 + \mu_2\, R_{ij}R^{ij} + \nu_1 R^3 + \nu_2 R R_{ij} R^{ij} \\ &\quad + \nu_3 R^i{}_j R^j{}_k R^k{}_i + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk} \nabla^i R^{jk} \end{aligned}$$

Phenomenological constraints from cosmology, PPN etc. restrict parameters

Renormalizability of projectable Hořava gravity I

Three key points for renormalizability:

1.) Finite set of counterterms

2.) Locality of counterterms

3.) Gauge invariance of counterterms

ad 1.) and 2.): (critical scaling $z = d$)

Anisotropic power counting: $\lim\limits_{b\to\infty} \{\omega \mapsto b^d\,\omega, \quad \mathbf{k} \mapsto b\,\mathbf{k}\}$

$$D_{\rm div} = 2\,d - d\,T - X - (d-1)\,l_{\vec{N}}, \quad D_{\rm div} < 0 \text{ diagram convergent}$$

Order by order subtraction of subdivergences works as in relativistic theory [Anselmi (2007)]. But even in absence of subdivergences extra difficulty due to spurious divergences in non-relativistic loop integrals

$$I = \int \mathrm{d}\omega_{(1)}\,\mathrm{d}^d k_{(1)} \underbrace{\int \prod_{l=2}^{L} [\mathrm{d}\omega_{(l)}\mathrm{d}^d k_{(l)}]\, f(\{\omega_{(l)}\},\{k_{(l)}\})}_{=\tilde{f}(\omega_{(1)},k_{(1)}) \text{ convergent}}, \quad [I] = D_{\rm div} < 0,$$

$$f(\omega_{(1)},k_{(1)}) = \omega_{(1)}^{-1+n}\,k_{(1)}^{-d-dn+D_{\rm div}} \quad \text{or} \quad f(\omega_{(1)},k_{(1)}) = \omega_{(1)}^{-1-n}\,k_{(1)}^{-d+dn+D_{\rm div}}$$

Renormalization of Hořava gravity II

For regular propagators $D_{\rm div} < 0$ really means convergence

$$\langle \phi_1, \phi_2 \rangle = \sum \frac{P(\omega, \mathbf{k})}{D(\omega, \mathbf{k})}, \quad D = \prod_m [A_m \, \omega^2 + B_m \, k^{2d} + ...]^{-1}, \quad A_m, B_m > 0,$$

$$[P] = r_1 + r_2 + 2(M-1)d, \qquad [\langle \phi_1(t, \mathbf{x}), \phi_2(0) \rangle] = r_1 + r_2 - 2d$$

Propagators can be brought into regular form by non-local gauge fixing

ad 3.):

Manifest FDiff covariant formulation for arbitrary backgrounds N, N_i, γ_{ij}

$$D_t = \frac{1}{N} \left(\partial_t - \mathcal{L}_{\vec{N}} \right), \quad \nabla_i = \partial_i + \Gamma_i(\gamma),$$

$$K_{ij} = 2 D_t \gamma_{ij}, \quad R_{ijkl}, \quad a_i = \nabla_i \ln N$$

For a large class of gauge theories, including Hořava gravity, counterterms are gauge invariant (Sergey's talk)

[Barvinsky, Blas, Herrero-Valea, Sibiryakov, CS (2017)]

Projectable Hořava gravity is perturbatively renormalizable

[Barvinsky, Blas, Herrero-Valea, Sibiryakov, CS (2016)]

Renormalizability of non-projectable Hořava gravity remains inconclusive – no regular gauge available

$2+1$ projectable Hořava gravity

Action for $D = 2+1$ projectable ($N=1$) Hořava gravity ($\Lambda = 0$),

$$S_{\rm HG} = \frac{1}{2\,G}\int {\rm d}t{\rm d}^2x\,\gamma^{1/2}\Big\{K_{ij}K^{ij} - \lambda\,K^2 + \mu\,R^2\Big\}$$

Gauge condition, Nielsen-Kallosh operator, gauge-fixing and ghost action

$$\chi^i = D_t n^i + \frac{1}{2\,\sigma}O^{ij}(\bar\nabla)\left(\bar\nabla^k h_{kj} - \lambda\bar\nabla_j h\right), \quad O^{ij} = -(\bar\gamma^{ij}\bar\Delta + \xi\bar\nabla^i\bar\nabla^j),$$

$$S_{\rm gf} = \int {\rm d}t\,{\rm d}x^2\,\gamma^{1/2}\,\chi^i O_{ij}\,\chi^j, \quad O_{ij} = -[\bar\gamma^{ij}\bar\Delta + \xi\bar\nabla^i\bar\nabla^j]^{-1},$$

$$S_{\rm gh} = \int {\rm d}t\,{\rm d}x^2\gamma^{1/2}\,c_i^*\,Q^i{}_j(\bar\nabla)\,c^j, \quad Q^i{}_j = \delta\chi^i[h^\varepsilon, n^\varepsilon]/\delta\varepsilon^j$$

Expand around arbitrary backgroud $\gamma_{ij} = \bar\gamma_{ij} + h_{ij}$, $N_i = \bar N_i + n_i$

$$S^{(2)}_{\rm tot} = S^{(2)} + S_{\rm gf} + S_{\rm gh}$$

Non-local terms only remain in shift-shift sector. Can be localized by "integrating in" auxiliary field π_i (cf. Nakanishi-Lautrup field in BRST)

$$\sigma\,(D_t n^i)\,O_{ij}\,(D_t n_j) \mapsto \frac{1}{2\sigma}\pi_i O^{ij}\pi_j - i\,\pi_i(D_t n^i)$$

Evaluation of diagrams

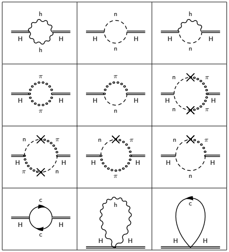
Regular gauge available in $2+1$ for which all propagators share the same pole

$$\mathcal{P}_{\rm s}(\omega,k)=\left[\omega^2+4\mu\frac{1-\lambda}{1-2\lambda}k^4\right]^{-1},\quad \sigma=\frac{1-2\lambda}{8\mu(1-\lambda)},\quad \xi=-\frac{1-2\lambda}{2(1-\lambda)}$$

Expand $\bar{\gamma}_{ij}=\delta_{ij}+H_{ij}$ and integrate out quantum fields h_{ij}, n_i, c_i, c_*^i

$$S_{\rm H}=\frac{1}{2G}\int {\rm d}t{\rm d}^2x\Big\{\frac{1}{4}\left(\dot{H}_{ij}\dot{H}^{ij}-\lambda\dot{H}\dot{H}\right)$$
$$-\mu\partial_k\partial_l H^{kl}(2\Delta H-\partial_j\partial_i H^{ij})+\mu\Delta H\Delta H+O(H^3)\Big\}$$

One-loop renormalization of G, λ and μ from two-point function of H_{ij}



Only λ and the combination $\mathcal{G}=G/\sqrt{\mu}$ are essential couplings

Integrate out quantum fluctuations h_{ij}, n_i, c_i c_*^i and collect logarithmic divergences $\rightarrow$ beta functions

One-loop RG flow of $2+1$ projectable Hořava gravity

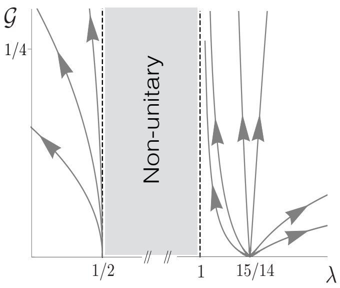
$$k_* \frac{\mathrm{d}\lambda}{\mathrm{d}k_*} = \beta_\lambda = \frac{15-14\lambda}{64\pi}\sqrt{\frac{1-2\lambda}{1-\lambda}}\mathcal{G},$$

$$k_* \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}k_*} = \beta_\mathcal{G} = -\frac{(16-33\lambda+18\lambda^2)}{64\pi\,(1-\lambda)^2}\sqrt{\frac{1-\lambda}{1-2\lambda}}\mathcal{G}^2$$

Two fixed points $(\lambda, \mathcal{G})$ at $(1/2, 0)$ and $(15/14, 0)$

$D = 2+1$ Hořava gravity is asymptotically free !

[Barvinsky, Blas, Herrero-Valea, Sibiryakov, CS, prepared for submission]



"Gravitational coupling" $\mathcal{G}$ asymptotically free in the UV and strongly coupled in the IR (in $d > 2$ relevant operators are present that cut off the flow)

In $D = 3+1$, asymptotic RG trajectory for which $|\lambda - 1| \to 0$ in IR might be interpreted as "accidental" low energy restoration of Lorentz invariance

Heat kernel & one-loop effective action

Covariant minimal $2_{\rm nd}$ order operator

$$\hat{F}(\nabla_X) = \Box + \hat{P} - \frac{\hat{I}R}{6} - m^2\,\hat{I}, \quad [\nabla_\mu, \nabla_\nu]\phi = \hat{\mathcal{R}}_{\mu\nu}\phi, \quad \Box = -g^{\mu\nu}\nabla_\mu\nabla_\nu$$

Heat kernel trace in D dimensions – asymptotic expansion $s \to 0$

$$\mathrm{Tr}\, e^{-s\,\hat{F}(\nabla_X)}\delta(X,Y) = \frac{1}{s^{D/2}}\sum_{n=0}^{\infty} s^n A_n, \quad A_n = \int \mathrm{d}^D X\, \frac{g^{1/2}(X)}{(4\pi)^{D/2}}\mathrm{tr}\,\hat{a}_n(X,X)$$

Schwinger-DeWitt representation for heat kernel of $\hat{F}$ available

$$e^{-s\hat{F}(\nabla_X)}\,\delta(X,Y) = \frac{\Delta(X,Y)}{(4\,\pi\,s)^{D/2}}\,g^{1/2}(Y)\,e^{-\frac{\sigma(X,Y)}{2\,s} - s\,m^2}\sum_n s^n\,\hat{a}_n(X,Y)$$

Divergent part of the one-loop effective action in $D = 2\omega = 4$ in closed form

$$\Gamma_1^{\rm div} = \frac{1}{2}\,\mathrm{Tr}\log\hat{F}(\nabla_X)\big|^{\rm div} = \lim_{\omega\to 2}\frac{1}{\omega/2 - 2}\,\frac{1}{32\,\pi^2}\int \mathrm{d}^4 X\, g^{1/2}(X)\,\mathrm{tr}\,\hat{a}_2(X,X)$$

$$\hat{a}_2(X,X) = \frac{1}{180}\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - R^{\mu\nu}R_{\mu\nu} + \Box\,R\right)\hat{I} + \frac{1}{2}\hat{P}^2 + \frac{1}{12}\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu} + \frac{1}{6}\Box\,\hat{P}$$

[DeWitt (1965), Gilkey (1984)]

Technique can be extended to higher derivative and non-minimal operators

[Barvinsky, Vilkovisky (1985)]

Reduction algorithm I

$$\mathcal{D} = \underbrace{\mathcal{A} + f(\mathcal{B})}_{\text{principle part}} + \underbrace{\mathcal{C}(D_t, \nabla)}_{\text{lower derivatives}}, \qquad (\mathcal{C} = 0 \text{ for illustration})$$

Step 1: Split kernel: Zassenhaus commutator insertions I

$$\mathrm{Tr}\, e^{-s\mathcal{D}} = \mathrm{Tr}\left[e^{-s(\mathcal{A}+f(\mathcal{B}))}\right] = \mathrm{Tr}\left[e^{-s\mathcal{A}} e^{-s f(\mathcal{B})} \mathcal{M}(-s\mathcal{A}, -s\mathcal{B})\right]$$

Step 2: Reduction of $f(B)$ to B: Resolvent method

$$e^{-s\, f(\mathcal{B})} = \frac{1}{2\pi\, i} \int_C \mathrm{d}q\, \frac{e^{-s\, f(q)}}{q - \mathcal{B}} = -\frac{1}{2\pi\, i} \int_C \mathrm{d}q\, e^{-s\, f(q)} \int_0^{i\, sgn(\Im\mathfrak{m} q)\infty} \mathrm{d}\lambda\, e^{\lambda(q-\mathcal{B})}$$

$$\mathrm{Tr}\, e^{-s\mathcal{D}} = -\int_C \frac{\mathrm{d}q}{2\pi\, i} \int_0^{i\, sgn(\Im\mathfrak{m} q)\infty} \mathrm{d}\lambda\; e^{-s\, f(q) + \lambda\, q}\, \mathrm{Tr}\left[e^{-sA} e^{-\lambda\mathcal{B}} \mathcal{M}(-s\mathcal{A}, -s\mathcal{B})\right]$$

Reduction algorithm II

Step 3: Recombine Kernels: Zassenhaus commutator insertions II

$$\text{Tr}\, e^{-s\mathcal{D}} = -\int_C \frac{\mathrm{d}q}{2\pi\, i} \int_0^{\pm i\,\infty} \mathrm{d}\lambda\; e^{-s\, f(q)+\lambda\, q}\; \text{Tr}\left[\Omega\; e^{-(s\mathcal{A}+\lambda\mathcal{B})}\right]$$

$$\Omega(s,\lambda; D_t, \nabla_x) := \mathcal{M}^{-1}(-s\mathcal{A}, -\lambda\mathcal{B})\,\mathcal{M}(-s\mathcal{A}, -s\mathcal{B}) = 1 + \mathcal{O}([\mathcal{A},\mathcal{B}])$$

Step 4: Define auxiliary metric $\tilde{g}_{\mu\nu}$ and reduce to $2_{\rm nd}$ order covariant operator

$$\tilde{\mathcal{A}}(s/\lambda; \tilde{\nabla}) := \frac{s}{\lambda}\mathcal{A} + \mathcal{B} = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu + \tilde{P}$$

Schwinger-DeWitt representation for covariant minimal $2_{\rm nd}$ order $\tilde{\mathcal{A}}$ known

$$\text{Tr}\left[\Omega(s,\lambda;\; D_t, \nabla_x)\, e^{-\lambda\tilde{\mathcal{A}}}\right] = \sum_k \text{Tr}\left[\mathfrak{B}_k^{i_1\ldots i_n}(s,\lambda)\nabla_{i_1}\ldots\nabla_{i_n} D_t\ldots D_t\, e^{-\lambda\tilde{\mathcal{A}}}\Big|_{Y=X}\right]$$

[Barvinsky, Blas, Herrero-Valea, Nesterov, Pérez-Nadal, CS (2017)]

Reduction algorithm for anisotropic operators

Case of anisotropic operator as in Lifshitz theories

$$\mathcal{A} = D_t^2, \quad \mathcal{B} = \Delta, \qquad f(\mathcal{B}) = \frac{\mathcal{B}^z}{M^{2z-1}}$$

Integral over auxiliary parameters q, λ can be done explicitly

$$\mathsf{Tr} e^{-s\,\mathcal{D}} = \frac{1}{\tau^{\frac{d+z}{2z}}} \sum_n^{\infty} \tau^{n/z}\, B_{2n}[\mathcal{D}]$$

The (integrated) coincidence limits of the Schwinger-DeWitt coefficients of the anisotropic operator $\mathcal{D}$ are

$$B_{2n}[\mathcal{D}] = \frac{1}{(4\pi)^{\frac{d+1}{2}}} \sum_{m=0}^{2n} M^{\frac{2n+dz-d}{z}-m} B_{m,2n}[\mathcal{D}]$$

Integrand of $B_{m,2n}$ is finite sum of geometrical invariants (build from D_t and ∇_i derivatives of K_{ij}, R_{ijkl} and a_i) of physcial dimensionality m

[Barvinsky, Blas, Herrero-Valea, Nesterov, Pérez-Nadal, CS (2017)]

Explicit calculation for Lifshitz operators straightforward but tedious

[Nesterov, Soludukhin (2011), D'Odorico, Goossens, Saueressig (2015)]

Conclusion and outlook

- Projectable Hořava gravity perturbatively renormalizable
  [Barvinsky, Blas, Herrero-Valea, Sibiryakov, CS (2016)]
  Phys. Rev. D **93** 064022 (2016), arXiv:1512.02250

- Counterterms are gauge invariant (FDiff invariant)
  [Barvinsky, Blas, Herrero-Valea, Sibiryakov, CS (2017)]
  arXiv:1705.03480

- Projectable Hořava gravity in $D = 2 + 1$ is asymptotically free
  [Barvinsky, Blas, Herrero-Valea, Sibiryakov, CS (2017)]
  prepared for submission

- Extension of heat kernel technique to anisotropic theories
  [Barvinsky, Blas, Herrero-Valea, Nesterov, Perez-Nadal, CS (2017)]
  arXiv:1703.04747

- To do: One-loop beta functions for (projectable) Hořava gravity (including matter) in $D = 3 + 1$ dimensions