Black Holes in Higher Derivative Gravity

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K.S.S., Gen.Rel.Grav. 9 (1978) 353
H. Lü, A. Perkins, C.N. Pope & K.S.S.,
PRL 114, 171601 (2015); arXiv 1502.01028
H. Lü, A. Perkins, C.N. Pope & K.S.S.,
Phys.Rev. D92 (2015) 12, 124019; arXiv 1508.00010
H. Lü, A. Perkins, C.N. Pope & K.S.S., arXiv 1704.05493

Quantum Context

One-loop quantum corrections to General relativity in 4-dimensional spacetime produce ultraviolet divergences of curvature-squared structure.

G. 't Hooft and M. Veltman, *Ann. Inst. Henri Poincaré* **20**, 69 (1974)

Inclusion of $\int d^4x\sqrt{-g}(\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)$ terms ab initio in the gravitational action leads to a renormalizable $D = 4$ theory, but at the price of a loss of *unitarity* owing to the modes arising from the $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ term, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

K.S.S., *Phys. Rev.* **D16**, 953 (1977).

[In $D = 4$ spacetime dimensions, this $(\text{Weyl})^2$ term is equivalent, up to a topological total derivative *via* the Gauss-Bonnet theorem, to the combination $\alpha(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)$].

Despite the apparent nonphysical behavior, quadratic-curvature gravities continue to be explored in a number of contexts:

- *Cosmology:* Starobinsky's original model for inflation was based on a $\int d^4x\sqrt{-g}(-R+\beta R^2)$ model.

  A.A. Starobinsky 1980; Mukhanov & Chibisov 1981

  This early model has been quoted (at times) as a good fit to CMB fluctuation data from the Planck satellite.

  J. Martin, C. Ringeval and V. Vennin, 1303.3787

- The *asymptotic safety scenario* considers the possibility that there may be a non-Gaussian renormalization-group fixed point and associated flow trajectories on which the ghost states arising from the $(\mathrm{Weyl})^2$ term could be absent.

  S. Weinberg 1976, M. Reuter 1996, M. Niedermaier 2009

Classical gravity with higher derivatives

Consider the gravitational action

$$I = \int d^4x\sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)\,.$$

The field equations following from this higher-derivative action are

$$\begin{aligned} H_{\mu\nu} \;=\; & \gamma\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{2}{3}(\alpha - 3\beta)\nabla_\mu\nabla_\nu R - 2\alpha\Box R_{\mu\nu} \\ & + \frac{1}{3}(\alpha + 6\beta)\, g_{\mu\nu}\Box R - 4\alpha R^{\eta\lambda}R_{\mu\eta\nu\lambda} + 2\left(\beta + \frac{2}{3}\alpha\right) RR_{\mu\nu} \\ & + \frac{1}{2}g_{\mu\nu}\left(2\alpha R^{\eta\lambda}R_{\eta\lambda} - \left(\beta + \frac{2}{3}\alpha\right) R^2\right) = \frac{1}{2}T_{\mu\nu} \end{aligned}$$

Linearized analysis of the vacuum ($T_{\mu\nu} = 0$) solutions of this theory about flat space reveals the following weak-field dynamical content:

- *positive-energy* massless spin-two
- *negative-energy* massive spin-two with mass $m_2 = \gamma^{\frac{1}{2}}(2\alpha)^{-\frac{1}{2}}$
- *positive-energy* massive spin-zero with mass $m_0 = \gamma^{\frac{1}{2}}(6\beta)^{-\frac{1}{2}}$

K.S.S. 1978

Nonlinear field equations for spherical symmetry

Use Schwarzschild coordinates

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

One might expect that the field equations for A and B would be of fourth order. However, there are some modest "gifts" arising from the diffeomorphism invariance of the theory, which gives rise to Bianchi-type identities for the field equations. One has

$$0 \equiv \nabla_\mu H^{\mu r} = \partial_r H^{rr} + \partial_0 H^{0r} + \partial_{\theta_\alpha} H^{\theta_\alpha r} + \Gamma^\mu_{\mu\nu} H^{\nu r} + \Gamma^r_{\mu\nu} H^{\mu\nu}$$

Since the second and subsequent terms here are of maximum fourth order in ∂_r derivatives, one learns that H^{rr} can be at most of third order in ∂_r derivatives. (Analogously in GR, G^{rr} is only of first order in ∂_r derivatives.)

The other field equations are indeed of fourth order in ∂_r derivatives. However, by ∂_r differentiating H^{rr} and taking an appropriate combination of this together with the other field equations, fourth-order ∂_r derivatives can be eliminated, giving a second third-order equation.

The remaining field equations are related to these two third-order equations by Bianchi-type identities. So the system that one actually has to solve for static and spherically symmetric spacetimes is one of two third-order ordinary differential equations in the radius r.

Accordingly, one expects a maximum of six integration constants to appear in the general solution.

Static and spherically symmetric solutions

Now consider spherically symmetric gravitational solutions in the linearised limit of the higher-curvature theory. In the linearized theory, one finds the following general solution to the source-free field equations $H^L_{\mu\nu} = 0$, in which $C, C^{2,0}, C^{2,+}, C^{2,-}, C^{0,+}, C^{0,-}$ are six integration constants:

$$A(r) =$$
$$1 - \frac{C^{20}}{r} - C^{2+}\frac{e^{m_2 r}}{2r} - C^{2-}\frac{e^{-m_2 r}}{2r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$
$$+\frac{1}{2}C^{2+}m_2 e^{m_2 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} - C^{0+}m_0 e^{m_0 r} + C^{0-}m_0 e^{-m_0 r}$$

$$B(r) =$$
$$C + \frac{C^{20}}{r} + C^{2+}\frac{e^{m_2 r}}{r} + C^{2-}\frac{e^{-m_2 r}}{r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$

- As one might expect from the dynamics of the linearized theory, the general static, spherically symmetric solution is a combination of a massless Newtonian $1/r$ potential plus rising and falling Yukawa potentials arising in both the spin-two and spin-zero sectors.
- When coupling to non-gravitational matter fields is made via standard $h^{\mu\nu}T_{\mu\nu}$ minimal coupling, one gets values for the integration constants from the specific form of the source stress tensor. Requiring asymptotic flatness and coupling to a point-source positive-energy matter delta function $T_{\mu\nu} = \delta^0_\mu \delta^0_\nu M \delta^3(\vec{x})$, for example, one finds

$$\begin{aligned} A(r) &= 1 + \frac{\kappa^2 M}{8\pi\gamma r} - \frac{\kappa^2 M(1+m_2 r)}{12\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M(1+m_0 r)}{24\pi\gamma}\frac{e^{-m_0 r}}{r} \\ B(r) &= 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma}\frac{e^{-m_0 r}}{r} \end{aligned}$$

with specific combinations of the Newtonian $1/r$ and falling Yukawa potential corrections arising from the spin-two and spin-zero sectors.

Frobenius Asymptotic Analysis

Asymptotic analysis of the field equations near the origin leads to study of the *indicial equations* for behavior as $r \to 0$. K.S.S. 1978
Let

$$\begin{aligned} A(r) &= a_s r^s + a_{s+1} r^{s+1} + a_{s+2} r^{s+2} + \cdots \\ B(r) &= b_t r^t + b_{t+1} r^{t+1} + b_{t+2} r^{t+2} + \cdots \end{aligned}$$

and analyze the conditions necessary for the lowest-order terms in r of the field equations $H_{\mu\nu} = 0$ to be satisfied. This gives the following results, for the general α, β higher derivative theory:

$$\begin{aligned} (s,t) &= (1,-1) && \text{with 5 free parameters} \\ (s,t) &= (0,0) && \text{with 3 free parameters} \\ (s,t) &= (2,2) && \text{with 6 free parameters} \end{aligned}$$

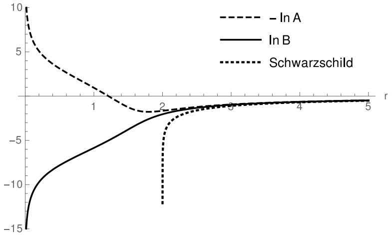
Lü, Perkins, Pope & K.S.S., 1508.00010

(2,2) solutions without horizons

For asymptotically flat solutions with nonzero spin-two Yukawa coefficient $C^{2-} \neq 0$, one finds numerical solutions that can continue on in to mesh with the (2,2) family obtained from Frobenius asymptotic analysis around the origin. Such solutions have no horizon; numerical solutions have been found in the $m_2 = m_0$ theory

B. Holdom, Phys.Rev. D66 (2002) 084010 and in the $R + C^2$ theory

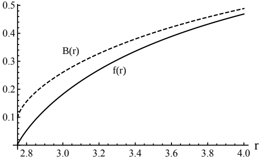
Lü, Perkins, Pope & K.S.S., 1508.00010



Horizonless solution in $R + C^2$ theory, behaving as r^2 in both $A(r)$ and $B(r)$ as $r \to 0$.

Wormholes

Another solution type found numerically has the character of a "wormhole". Such solutions can have either sign of $M \sim -C^{20}$ and either sign of the falling Yukawa coefficient C^{2-}. As an example, one finds a solution with $M < 0$ in the $R - C^2$ theory



In this solution, $f(r) = 1/A(r)$ reaches zero at a point where $B(r) = a_0^2 > 0$. Making a coordinate change $r - r_0 = \frac{1}{4}\rho^2$, one then has

$$ds^2 = -(a_0^2 + \tfrac{1}{4}B'(r_0)\rho^2)dt^2 + \frac{d\rho^2}{f'(r_0)} + (r_0^2 + \tfrac{1}{2}r_0\rho^2)d\Omega^2$$

which is $\mathbb{Z}_2$ symmetric in ρ and can be interpreted as a "wormhole", with the $r < r_0$ region excluded from spacetime.

Black hole solutions

If one assumes the existence of a horizon and assumes also asymptotic flatness at infinity, then a no-hair theorem for the trace of the field equations implies that the Ricci scalar must vanish: $R = 0$. W. Nelson, 1010.3986; H. Lü, A. Perkins, C.N. Pope & K.S.S., 1508.00010 This significantly simplifies the analysis of the solutions. The field equations then become identical to those in the $\beta = 0$ case, *i.e.* with just a $(\mathrm{Weyl})^2$ term and no R^2 term in the action.

Counting parameters in an expansion around the horizon, subject to the $R = 0$ condition, one finds just 3 free parameters. This is the same count as in the (1,-1) family of an expansion around the origin when subjected to the $R = 0$ condition. So asymptotically flat solutions with a horizon must belong to the (1-1) family, which contains the Schwarzschild solution itself. The Schwarzschild solution is characterized by two parameters: the mass M of the black hole, plus a trivial g_{00} normalization at infinity. So in the higher-derivative theory, there is just one "non-Schwarzschild" (1,-1) parameter.

Away from Schwarzschild in the (1,-1) family

Considering variation of this "non-Schwarzschild" parameter away from the Schwarzschild value, it is clear that changing it generally has to do something to the solution at infinity. For a solution assumed to have a horizon, and holding $R = 0$, the only thing that can happen initially is that the rising exponential is turned on, *i.e.* asymptotic flatness is lost. So, for asymptotically flat solutions with a horizon *in the vicinity of the Schwarzschild solution*, the only spherically symmetric static solution generally is Schwarzschild itself.

One concludes that the Schwarzschild black hole is at least in general *isolated* as an asymptotically flat solution with a horizon.

Non-Schwarzschild Black Holes Lü, Perkins, Pope & K.S.S., 1508.00010; 1502.01028

Now the question arises: what happens when one moves a finite distance away from Schwarzschild in terms of the (1,-1) non-Schwarzschild parameter? Does the loss of asymptotic flatness persist, or does something else happen, with solutions arising that cannot be treated by a linearized analysis in deviation from Schwarzchild?

This can be answered numerically. In consequence of the trace no-hair theorem, the assumption of a horizon together with asymptotic flatness requires $R = 0$ for the solution, so the calculations can effectively be done in the $R - C^2$ theory with $\beta = 0$, in which the field equations, thankfully, can be reduced to a system of two second-order equations.

The study of non-Schwarzschild solutions is more easily carried out with a metric parametrization

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

i.e. by letting $A(r) = 1/f(r)$.

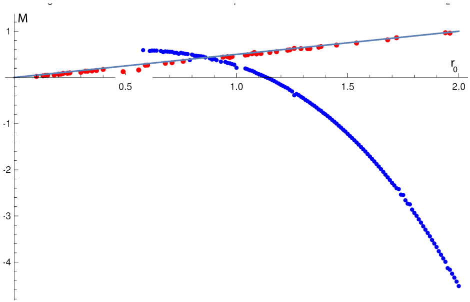
For $B(r)$ vanishing linearly in $r - r_0$ for some r_0, analysis of the field equations shows that one must then also have $f(r)$ similarly linearly vanishing at r_0, and accordingly one has a horizon. One can thus make near-horizon expansions

$$\begin{aligned} B(r) &= c\left[(r-r_0) + h_2\,(r-r_0)^2 + h_3\,(r-r_0)^3 + \cdots\right] \\ f(r) &= f_1\,(r-r_0) + f_2\,(r-r_0)^2 + f_3\,(r-r_0)^3 + \cdots \end{aligned}$$

and the parameters h_i and f_i for $i \geq 2$ can then be solved-for in terms of r_0 and f_1. For the Schwarzschild solution, one has $f_1 = 1/r_0$, so it is convenient to parametrize the deviation from Schwarzschild using a non-Schwarzschild parameter δ with

$$f_1 = \frac{1+\delta}{r_0}\,.$$

The task then becomes that of finding values of $\delta \neq 0$ for which the generic rising exponential behavior as $r \to \infty$ is suppressed. What one finds is that there does indeed exist an asymptotically flat family of non-Schwarzschild black holes which crosses the Schwarzschild family at a special horizon radius r_0^{Lich}. For $\alpha = \frac{1}{2}$, one finds the following phases of black holes:



Black-hole masses as a function of horizon radius r_0, with a crossing point at $r_0^{\mathrm{Lich}} \simeq 0.876$. The red family denotes Schwarzschild black holes and the blue family denotes non-Schwarzschild black holes.

The Lichnerowicz Operator

Now let us study in some more detail the point where the new black hole family crosses the classic Schwarzschild solution family. We can study solutions in the vicinity of the Schwarzschild family by looking at infinitesimal variations of the higher-derivative equations of motion around a Ricci-flat background. For the $\delta R_{\mu\nu}$ variation of the Ricci tensor away from a background with $R_{\mu\nu} = 0$ one obtains

$$\gamma(\delta R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, \delta R) + 2(\beta - \tfrac{1}{3}\alpha)(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu)\delta R$$
$$-2\alpha\Box\,(\delta R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, \delta R) - 4\alpha\, R_{\mu\rho\nu\sigma}\, \delta R^{\rho\sigma} = 0 \ .$$

Restricting attention to asymptotically flat solutions with horizons, however, we know from the trace no-hair theorem that $R = 0$ so $\delta R = 0$ and the $\delta R_{\mu\nu}$ equation simplifies, upon recalling that $m_2^2 = \frac{\gamma}{2\alpha}$, to

$$\left(\Delta_L + m_2^2\right) \delta R_{\mu\nu} = 0 \ ,$$

where the Lichnerowicz operator is given by

$$\Delta_L \, \delta R_{\mu\nu} \equiv -\Box \delta R_{\mu\nu} - 2 R_{\mu\rho\nu\sigma} \, \delta R^{\rho\sigma} \ .$$

Restricting attention to the $m_2^2 > 0$ nontachyonic case, one sees that black hole solutions deviating from Schwarzschild must have a $\lambda = -m_2^2$ negative Lichnerowicz eigenvalue for $\delta R_{\mu\nu}$.

The Gross-Perry-Yaffe eigenvalue

In a study of the thermodynamic instability of the Euclideanised Schwarzschild solution in Einstein theory, Gross, Perry and Yaffe Phys.Rev. D25 (1982), 330 found that there is just one normalisable negative-eigenvalue mode of the Lichnerowicz operator for deviations from the Schwarzschild solution. For a Schwarzschild solution of mass M, it is

$$\begin{aligned} \lambda &\simeq -0.192M^{-2} \\ \textit{i.e.} \quad m_2 M &\simeq 0.438 \simeq \sqrt{.192} \end{aligned}$$

- Comparing with the numerical results for the new black hole solutions of the higher-derivative gravity theory, this corresponds nicely with the point where the new black hole family crosses the Schwarzschild family.

Time Dependent Solutions and Stability

Now consider *time-dependent* perturbations $\delta R_{\mu\nu}$ away from a Schwarzschild solution in order to search for possible instabilities within the higher-derivative theory. For this one needs to analyse the Lichnerowicz condition $(\Delta_L + m_2^2)\,\delta R_{\mu\nu} = 0$ for time-dependent solutions. For asymptotically flat solutions with a horizon, we still have the $R = 0$ consequence of the trace no-hair theorem, so $\delta R = 0$. Then from the Bianchi identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$ we obtain $\nabla^\mu \delta R_{\mu\nu} = 0$, so $\delta R_{\mu\nu}$ must be a "TT" quantity.

The "TT" condition for $\delta R_{\mu\nu}$ already indicates a similarity to the situation that obtains in Pauli-Fierz theory, where the linearised field equations for a massive spin-two field $\psi_{\mu\nu}$ imply $\partial^\mu \psi_{\mu\nu} = \psi^\nu{}_\nu = 0$.

Gregory-Laflamme Instability

Analysis of the possibility of growing $(\mathrm{Re}(\nu) > 0)$ perturbations can be approached using WKB methods B.F. Schutz and C.M. Will, Ap.J. 291:L33 (1985) or numerically. But in fact, the answer has been known for some time from the 5D string R. Gregory and R. Laflamme, PRL 70 (1993) 2837 . Considering perturbations about the 5D black string $ds^2_{(5)} = ds^2_{(4)} + dz^2$

$$\begin{pmatrix} h^{(4)}_{\mu\nu} & h_{\mu z} \\ h_{z\nu} & h_{zz} \end{pmatrix} \tag{1}$$

where the z dependence is assumed to be of the form $e^{\mathrm{i}kz}$, one finds that $h^{(4)}_{\mu\nu}$ satisfies an equation of the same Lichnerowicz form $(\Delta_L + k^2)\, h^{(4)}_{\mu\nu} = 0$ as for $\delta R_{\mu\nu}$ Y.S. Myung, Phys.Rev. D88 (2013) . This form is also found for perturbations about the Schwarzschild solution in dRGT nonlinear massive gravity.

The Gregory-Laflamme instability is an S-wave ($\ell = 0$) spherically symmetric instability from the 4D perspective. In the higher-derivative theory, it exists for low-mass Schwarzschild black holes, which disappears for black hole masses $M \geq M_{\max}$ where

$$\frac{m_2 M_{\max}}{M_{\rm Pl}^2} = .438$$

This is precisely the crossing point between the family of new black holes and the Schwarzschild family.

Note that this monopole instability depends on the presence in the theory of the m_2 massive spin-two mode.

Thermodynamic Implications for Instability

The $D = 4$ Wald entropy formula

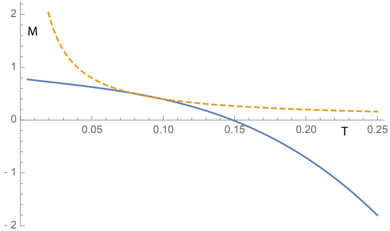
$$S = -\tfrac{1}{8}\int_{+}\sqrt{h}d\Sigma\,\epsilon^{ab}\epsilon^{cd}\frac{\partial L}{\partial R^{abcd}}$$

gives results that respect the first law of black-hole thermodynamics, $dM = TdS$.

For non-Schwarzschild black holes in $D = 4$, one obtains the following numerical relations between mass, temperature and entropy:

$$\begin{aligned} M_{\rm NSch} &\approx 0.168 + 0.131\,S - 0.00749\,S^2 - 0.000139\,S^3 + \cdots \\ T_{\rm NSch} &\approx 0.131 - 0.0151\,S - 0.000428\,S^2 + \cdots \end{aligned}$$

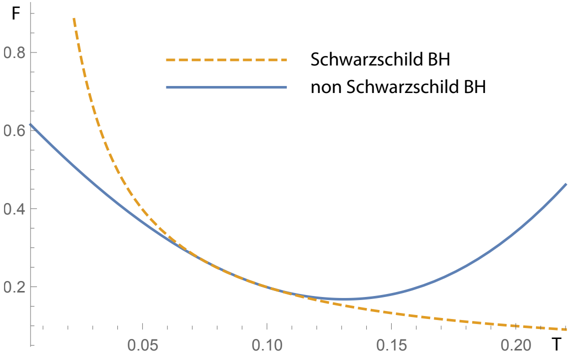
Recall that for Schwarzschild black holes, one has the classic mass-temperature relation $M_{\text{Sch}} = \frac{1}{8\pi T}$. Eliminating the entropy for the non-Schwarzschild black holes, one obtains the corresponding relations between black-hole mass and temperature for Schwarzschild and non-Schwarzschild black holes:



Mass versus temperature relations for Schwarzschld (dashed red) and non-Schwarzschild (solid blue) black holes.

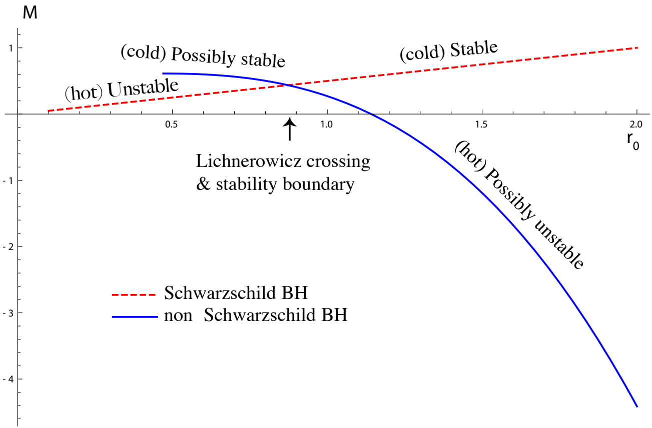
Free Energy

Consequently, for the free energy $F = M - TS$, one has the following situation, showing a switchover between the Schwarzschild and non-Schwarzschild solutions:



Free energy for Schwarzschild (dashed red) and non-Schwarzschild (blue) black holes. Lower free energy corresponds to greater stability.

We therefore have the following suggested stability picture:



Classical stability regimes. The dashed red line denotes Schwarzschild black holes and the solid blue line denotes non-Schwarzschild black holes.

Outlook

- Taking the fourth order field equations seriously for gravity including quadratic curvatures in the action leads to a rich space of asymptotically flat solutions including horizonless solutions, wormholes and both Schwarzschild and non-Schwarzschild black hole solutions.
- The branch of non-Schwarzschild black holes bears an intimate relation to black-hole stability: the branching point mass on the Schwarzschild family is also the upper limit of classical instability of the Schwarzschild solution.
- Thermodynamic considerations similar to the conjecture of Gubser and Mitra JHEP 0108 (2001) 018 on the relationship between thermodynamic and time-dependent dynamical instabilities suggest that there is a switchover of stability to the non-Schwarzschild branch of black-hole solutions for black holes with radii smaller than the Lichnerowicz crossing point.