# $N = 4$ supersymmetric multiparticles system on the conformally flat manifold

Anton Sutulin

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
Dubna, Russia

*in collaboration with Nikolay Kozyrev, Sergey Krivonos, Olaf Lechtenfeld, Armen Nersessian*
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# Plan

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The interest to the $N = 4$ supersymmetric mechanics as well as its superconformal versions has a long story. These one dimensional models have a deep connection with integrable models and also in the context of the Black holes. There are different ways of constructing the superextension of the bosonic systems. One of the most effective is based on the superfiled approach. Depending on the choice of the $N = 4$ supermultiplet, the corresponding action describes one of the following geometries of the target space:

- real Kahler for (1,4,3) multiplet
- Kahler for (2,4,2) multiplet
- hyper Kahler for (4,4,0) multiplet

The second approach consists in a direct construction of supercharges and Hamiltonian.

In my talk, I will follow the second approach to construct $N = 4$ mechanics on the conformally flat manifolds.

In the beginning, let me recall some basic results obtained in paper by S. Krivonos, O.Lechtenfeld (hep-th/1012.4639). There the authors considered the following Ansatz for the supercharges

$$Q^a = p_i\,\psi_i^a + \mathrm{i}W_i\,\psi_i^a + \mathrm{i}F_{ijk}\,\psi_i^b\,\psi_{bj}\,\bar\psi_k^a\,, \quad \overline{Q}_a = p_i\,\bar\psi_{ia} - \mathrm{i}W_i\,\bar\psi_{ai} + \mathrm{i}F_{ijk}\,\bar\psi_{mb}\,\bar\psi_j^b\,\psi_{ka}, \qquad (1)$$

where $F_{ijk}$ is the totally symmetric function. The requirement that these supercharges form the $N = 4$ one dimensional superalgebra

$$\{Q^a, \overline{Q}_b\} = \frac{\mathrm{i}}{2}\,\delta_b^a\,H \qquad (2)$$

leads to the differential equations on functions $W_i$ and $F_{ijk}$ one of which was the WDVV equation

$$F_{ijk}F_{kmn} - F_{njk}F_{kmi} = 0.\partial_i F_{kmn} - \partial_k F_{imn} = 0 \quad \Rightarrow \quad F_{ijk} = \partial^3_{jki}F\,, \qquad (3)$$

and the symmetric function $F_{ijk}$ is satisfied the relation

$$\partial_i F_{kmn} - \partial_k F_{imn} = 0 \quad \Rightarrow \quad F_{ijk} = \partial^3_{jki}F\,. \qquad (4)$$

They also found many interesting solutions to this equation and constructed the corresponding supersymmetric mechanics.

In my talk, I will consider more general case which correspond to the $N = 4$ supersymmetric multiparticles system on the conformally flat manifold. One of the result of our consideration, it will be obtaining the generalization of the WDVV equations

We consider $n$-particles on a real line with bosonic coordinates and momenta $(x_i, p_i)$ as well as associated complex fermionic variables $(\psi_i^a, \bar\psi_i^a)$.
The indices $i, j, k$ run from 1 to $n$, while the indices $a, b, c$ are the spinor $su(2)$ indices: $a = 1, 2$.
Let us start from the following most general Ansatz for the supercharges:

$$
\begin{aligned}
Q^a &= f\, p_i\, \psi_i^a + \mathrm{i} W_i\, \psi_i^a + \mathrm{i} F_{ijk}\, \psi_i^b\, \psi_{bj}\, \bar\psi_k^a + \mathrm{i} G_{ijk}\, \psi_i^a\, \psi_j^b\, \bar\psi_{kb}\,, \\
\overline{Q}_a &= f\, p_i\, \bar\psi_{ia} - \mathrm{i} W_i\, \bar\psi_{ai} + \mathrm{i} F_{ijk}\, \bar\psi_{mb}\, \bar\psi_j^b\, \psi_{ka} + \mathrm{i} G_{ijk}\, \bar\psi_{ia}\, \bar\psi_{jb}\, \psi_k^b\,.
\end{aligned}
\tag{5}
$$

Here, $f, W_i, F_{ijk}$ and $G_{ijk}$ are arbitrary, for the time being functions depending on $n$-coordinates $x_i$. In addition, we assume that the functions $F_{ijk}$ and $G_{ijk}$ are symmetric and anti-symmetric over the first two indices, respectively:

$$
F_{ijk} = F_{jik}\,, \qquad G_{ijk} = -G_{jik}\,. \tag{6}
$$

The difference from the structure of the supercharges corresponded to the flat case is the presence of additional functions $f$ and $G_{ijk}$ in (5).

The Poisson brackets between the variables read

$$\{x_i, p_j\} = \delta_{ij}, \quad \{\psi_i^a, \bar{\psi}_{bj}\} = \frac{\mathrm{i}}{2}\delta_b^a\, \delta_{ij}. \tag{7}$$

The requirements that the supercharges $Q^a, \overline{Q}^b$ span $N = 4$ super-Poincaré algebra

$$\left\{Q^a, Q^b\right\} = 0, \ \left\{\overline{Q}_a, \overline{Q}_b\right\} = 0, \qquad \left\{Q^a, \overline{Q}_b\right\} = \frac{\mathrm{i}}{2}\,\delta_b^a\, H \tag{8}$$

will result in the following conditions on the functions involved:

$$G_{ijk} = \delta_{ik}\partial_j f - \delta_{jk}\partial_i f, \tag{9}$$

$$f\left(\partial_j W_i - \partial_i W_j\right) = G_{ijk} W_k, \tag{10}$$

$$F_{ijk} - F_{ikj} = \frac{3}{2} G_{jki}, \tag{11}$$

$$f\left(\partial_i W_j + \partial_j W_i\right) + \frac{1}{2} W_l\left(G_{lij} + G_{lji}\right) + W_l\left(F_{lij} + F_{lji}\right) = 0, \tag{12}$$

$$Y_{ijkl} - Y_{ikjl} = 0, \tag{13}$$

where

$$Y_{ijkl} = -2f\,\partial_k F_{ijl} - 2F_{ijm}F_{mkl} + F_{ijm}G_{kml} - F_{mil}G_{kjm} - F_{mjl}G_{kim} + f\,\partial_k G_{jil} - \frac{1}{4}\partial_i f\, G_{jkl}. \tag{14}$$

The first equation

$$G_{ijk} = \delta_{ik}\partial_j f - \delta_{jk}\partial_i f \tag{15}$$

completely defined the function $G_{ijk}$ in terms of the derivatives of the function $f$, while the second one

$$f\,(\partial_j W_i - \partial_i W_j) = G_{ijk} W_k \tag{16}$$

can be rewritten as

$$f^2\left[\partial_i\left(\frac{W_j}{f}\right) - \partial_j\left(\frac{W_i}{f}\right)\right] = 0 \quad \Rightarrow \quad W_i = f\,\partial_i W. \tag{17}$$

Here, $W$ is a new scalar function which defined the vector function $W_i$.
The third equation

$$F_{ijk} - F_{ikj} = \frac{3}{2} G_{jki} \tag{18}$$

fixes the antisymmetric part of the function $F_{ijk}$.

The function $F_{ijk}$ itself can be represented as

$$F_{ijk} = \frac{1}{3}\left(F_{ijk} + F_{ikj} + F_{jki}\right) + \frac{1}{3}\left(F_{ijk} - F_{ikj}\right) + \frac{1}{3}\left(F_{jik} - F_{jki}\right) \qquad (19)$$
$$= F^S_{ijk} - \frac{1}{2}G_{kji} - \frac{1}{2}G_{kij},$$

where, $F^S_{ijk}$ is defined to be fully symmetric over the indices:

$$F^S_{ijk} \equiv \frac{1}{3}\left(F_{ijk} + F_{ikj} + F_{jki}\right) \qquad (20)$$

and we used the equation (11) for the last two terms in (19). Thus, only this symmetric over the indices function $F^S_{ijk}$ is left undefined yet.

Now, plugging the expression

$$F_{ijk} = F^S_{ijk} - \frac{1}{2} G_{kji} - \frac{1}{2} G_{kij} \tag{21}$$

into the equation

$$Y_{ijkl} - Y_{ikjl} = 0, \tag{22}$$

we will get two equations - for symmetric and anti-symmetric parts over indices $(k, l)$:

$$\begin{aligned} & f \left( \partial_i F^S_{jkl} - \partial_j F^S_{ikl} \right) + \partial_j f\, F^S_{ikl} - \partial_i f\, F^S_{jkl} \\ & + \partial_m f \, \left( \delta_{jl} F^S_{ikm} + \delta_{jk} F^S_{ilm} - \delta_{il} F^S_{jkm} - \delta_{ik} F^S_{jlm} \right) = 0, \end{aligned} \tag{23}$$

$$f \left( \partial_i G_{lkj} - \partial_j G_{lki} \right) + F^S_{ikm} F^S_{jlm} - F^S_{jkm} F^S_{ilm} + \partial_m f\, \partial_m f \left( \delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl} \right) = 0. \tag{24}$$

Note that the terms $F \cdot F - F \cdot F$ in (24) looks like an expression corresponded those in the well-known WDVV equation.

The equation (12) can be also slightly simplified to be

$$f\left(\partial_i W_j + \partial_j W_i\right) + W_m\left(G_{mij} + G_{mji}\right) + 2 W_m F^S_{ijm} = 0. \tag{25}$$

Thus, the equations (23), (24) and (25), together with (9), (17), i.e.

$$G_{ijk} = \delta_{ik}\partial_j f - \delta_{jk}\partial_i f, \qquad W_i = f\partial_i W, \tag{26}$$

form the full set of equations for the supercharges (5), which span $N = 4$ super-Poincaré algebra (8).
Note that the bosonic part of the corresponding Hamiltonian reads

$$H_{bos} = \sum\left(f^2 p_i p_i + W_i W_i\right) = f^2 \sum\left(p_i p_i + \partial_i W \partial_i W\right). \tag{27}$$

The full Hamiltonian included spinor variables will be given later.

Let us present the another possibility of constructing $N = 4$ supercharges which has been proposed by S.Krivonos and O.Lechtenfeld.
It consists in the introducing of one set of additional (bosonic) spin variables $\{u^a, \bar{u}_a | a = 1, 2\}$ parametrizing an internal two-sphere and obeying the brackets

$$\left\{u^a, \bar{u}_b\right\} = -\mathrm{i}\,\delta^a_b. \tag{28}$$

These spin variables may be utilized to slightly modify the Ansatz (5) to be

$$\begin{aligned} Q^a &= f\,p_i\,\psi^a_i + U_i\,J^{ac}\,\psi_{ci} + \mathrm{i}\,F_{ijk}\,\psi^c_i\,\psi_{cj}\,\bar{\psi}^a_k + \mathrm{i}\,G_{ijk}\,\psi^a_i\,\psi^c_j\,\bar{\psi}_{ck}\,, \\ \overline{Q}_a &= f\,p_i\,\bar{\psi}_{ia} - U_i\,J_{ac}\,\bar{\psi}^c_i + \mathrm{i}\,F_{ijk}\,\bar{\psi}_{ci}\,\bar{\psi}^c_j\,\psi_{ak} - \mathrm{i}\,G_{ijk}\,\bar{\psi}_{ai}\,\psi^c_k\,\bar{\psi}_{cj}\,, \end{aligned} \tag{29}$$

with the $su(2)$ currents

$$J^{ab} = \frac{\mathrm{i}}{2}\left(u^a\bar{u}^b + u^b\bar{u}^a\right) \quad \Rightarrow \quad \left\{J^{ab}, J^{cd}\right\} = -\epsilon^{ac}J^{bd} - \epsilon^{bd}J^{ac}. \tag{30}$$

The spin variables just serve to produce these currents and do not appear by themselves.

The $N = 4$ super-Poincaré algebra (8) defines the Hamiltonian $H$ and enforces on the functions $F_{ijk}$ and $G_{ijk}$ the same conditions

$$
\begin{aligned}
& f\left(\partial_i F^S_{jkl} - \partial_j F^S_{ikl}\right) + \partial_j f\, F^S_{ikl} - \partial_i f\, F^S_{jkl} \\
& + \partial_m f\, \left(\delta_{jl} F^S_{ikm} + \delta_{jk} F^S_{ilm} - \delta_{il} F^S_{jkm} - \delta_{ik} F^S_{jlm}\right) = 0, \\
& f\,(\partial_i G_{lkj} - \partial_j G_{lki}) + F^S_{ikm} F^S_{jlm} - F^S_{jkm} F^S_{ilm} + \partial_m f\, \partial_m f\, (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) = 0.
\end{aligned}
$$

while the equations (25) and (26) changed to be

$$
f\left(\partial_i \partial_j U - \partial_i U\, \partial_j U\right) + \partial_i f\, \partial_j U + \partial_j f\, \partial_i U - \delta_{ij} \partial_k f\, \partial_k U - F^S_{ijk} U_k = 0, \qquad (31)
$$

$$
G_{ijk} = \delta_{ik} \partial_j f - \delta_{jk} \partial_i f, \quad U_i = f \partial_i U. \qquad (32)
$$

Finally note, that the bosonic part of the corresponding Hamiltonian reads

$$
H_{bos} = \sum \left(f^2 p_i p_i + \frac{1}{2} J^{ab} J_{ab}\, U_i U_i\right) = f^2 \sum \left(p_i p_i + \frac{1}{2} J^{ab} J_{ab}\, \partial_i U \partial_i U\right). \qquad (33)
$$

Now let us give the covariant form of the constraints which were considered in the previous section of my talk. The equations for the functions

$$f\left(\partial_i F^S_{jkl} - \partial_j F^S_{ikl}\right) + \partial_j f\, F^S_{ikl} - \partial_i f\, F^S_{jkl}$$
$$+\partial_m f\, \left(\delta_{jl} F^S_{ikm} + \delta_{jk} F^S_{ilm} - \delta_{il} F^S_{jkm} - \delta_{ik} F^S_{jlm}\right) = 0,$$
$$f\left(\partial_i G_{lkj} - \partial_j G_{lki}\right) + F^S_{ikm} F^S_{jlm} - F^S_{jkm} F^S_{ilm} + \partial_m f\, \partial_m f\, (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) = 0.$$

may be written in the geometrical form if we define the metric on the configuration space and the corresponding covariant objects (connection, Riemannian tensor):

$$g_{ij} = \frac{1}{f^2}\delta_{ij},$$
$$\Gamma^k_{ij} = \frac{1}{2}(g^{-1})^{km}\left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right) = -\frac{1}{f}\left(\partial_j f\, \delta^k_i + \partial_i f\, \delta^k_j - \delta^{km}\partial_m f\, \delta_{ij}\right),$$
$$R_{imjk} = g_{ii'} R^{i'}_{mjk} = \frac{1}{f^3}\left(\delta_{ij}\partial^2_{km} f - \delta_{ik}\partial^2_{jm} f + \delta_{mk}\partial^2_{ij} f - \delta_{mj}\partial^2_{ik} f\right)$$
$$+\frac{1}{f^4}\left(\delta_{ik}\delta_{jm} - \delta_{ij}\delta_{km}\right)\delta^{ps}\partial_p f\,\partial_s f,$$
$$\nabla_m F^S_{ijk} = \partial_m F^S_{ijk} - F^S_{ijn}\Gamma^n_{km} - F^S_{ink}\Gamma^n_{jm} - F^S_{njk}\Gamma^n_{im}. \qquad (34)$$

After rescaling the function $F^S_{ijk}$

$$\widetilde{F}_{ijk} = \frac{1}{f^3} F^S_{ijk}, \tag{35}$$

the equations (23) and (24) acquire the form

$$\nabla_i \widetilde{F}_{jkl} - \nabla_j \widetilde{F}_{ikl} = 0, \tag{36}$$

$$\widetilde{F}_{ik}{}^p \widetilde{F}_{jmp} - \widetilde{F}_{jk}{}^p \widetilde{F}_{imp} = R_{kmji}, \tag{37}$$

where

$$\widetilde{F}_{ij}{}^p = g^{pk} \widetilde{F}_{mjk}.$$

Thus, we recognize in the equations (37) the covariant form of WDVV equations. Using the equations in (36), it is an easy to check that the left hand size of (37) satisfies the Bianchi identity.

Finally, the covariant form of the equations (25) and (31) read:

$$(\nabla_i \nabla_j + \nabla_j \nabla_i)\, W + 2\nabla^k W\, \widetilde{F}_{ijk} = 0, \tag{38}$$

$$(\nabla_i \nabla_j + \nabla_j \nabla_i)\, U - 2\nabla_i U\, \nabla_j U - 2\nabla^k U\, \widetilde{F}_{ijk} = 0. \tag{39}$$

It is clear, that the solutions of the equations (38) and (39) are related as follow

$$W \;\Rightarrow\; e^{-U}, \qquad \widetilde{F}_{ijk} \;\Rightarrow\; -\widetilde{F}_{ijk}. \tag{40}$$

The solution of the system of equations (37) is very difficult to find explicitly.
For this reason, we focus the most attention on the two-dimensional case, where one can make use of three algebraic equations with potential (38), (39) and the single algebraic equation without potential (37) to find the four independent $\widetilde{F}_{ijk}$.
Then three remaining differential equations on $\widetilde{F}_{ijk}$ can be just checked to be satisfied.

Now let us give an explicit structure of the full Hamiltonian which is included the fermionic terms. To make the Hamiltonians simpler, we may introduce notation

$$\tilde{p}_k = p_k - \frac{2\mathrm{i}}{f} G_{ijk} \psi_i^c \bar{\psi}_{cj}. \tag{41}$$

Then all derivatives of $f$ can be hidden in geometric structures. In the case without harmonics, the Hamiltonian can be written as

$$\begin{aligned} H_W &= f^2 \tilde{p}_i \tilde{p}_i + f^2 \partial_i W \partial_i W + 4 f^2 \nabla_i \partial_j W \psi_i^c \bar{\psi}_{cj} \\ &- 2 f^4 \left( \nabla_i \widetilde{F}_{kmj} + \nabla_m \widetilde{F}_{ijk} + 2 R_{imjk} \right) \psi_i^c \bar{\psi}_{cm} \psi_j^d \bar{\psi}_{dk}. \end{aligned} \tag{42}$$

The Hamiltonian with harmonics is

$$\begin{aligned} H_J &= f^2 \tilde{p}_i \tilde{p}_i + \frac{1}{2} f^2 J_{cd} J^{cd} \partial_i U \partial_i U - 4\mathrm{i} f^2 \nabla_i \partial_j U J_{cd} \psi_i^c \bar{\psi}_j^d \\ &- 2 f^4 \left( \nabla_i \widetilde{F}_{kmj} + \nabla_m \widetilde{F}_{ijk} + 2 R_{imjk} \right) \psi_i^c \bar{\psi}_{cm} \psi_j^d \bar{\psi}_{dk}. \end{aligned} \tag{43}$$

The bosonic part of the Hamiltonians

$$H_{bos} = \sum \left( f^2 p_i p_i + W_i W_i \right) = f^2 \sum (p_i p_i + \partial_i W \partial_i W) \qquad (44)$$

and

$$H_{bos} = \sum \left( f^2 p_i p_i + \frac{1}{2} J^{ab} J_{ab} \, U_i U_i \right) = f^2 \sum \left( p_i p_i + \frac{1}{2} J^{ab} J_{ab} \, \partial_i U \partial_i U \right). \qquad (45)$$

coincide with the spherical part of the Calogero models if the metrics $g_{ij} = 1/f^2 \, \delta_{ij}$ defines the sphere. Thus, our supercharges provide the $N = 4$ supersymmetric extension of these spherical parts.

In the case of two particles with the metrics $f = e^{-x_1}$ the Hamiltonians (44), (45) describe the bosonic sector of 3-particles conformal mechanics with translation invariance (upon decouple of the center of mass).

Clearly, for the all $N = 4$ 3-particles Calogero models the supercharges introduced in the beginning of my talk provide a proper supersymmetric extension (with the proper choice of the potentials).

In the two-particle case one may solve the equations

$$\nabla_i \widetilde{F}_{jkl} - \nabla_j \widetilde{F}_{ikl} = 0, \tag{46}$$

$$\widetilde{F}_{ik}{}^{p} \widetilde{F}_{jmp} - \widetilde{F}_{jk}{}^{p} \widetilde{F}_{imp} = R_{kmji}, \tag{47}$$

$$f \left( \partial_i W_j + \partial_j W_i \right) + W_m \left( G_{mij} + G_{mji} \right) + 2 W_m F^S_{ijm} = 0 \tag{48}$$

or

$$f \left( \partial_i \partial_j U - \partial_i U \, \partial_j U \right) + \partial_i f \, \partial_j U + \partial_j f \, \partial_i U - \delta_{ij} \partial_k f \, \partial_k U - F^S_{ijk} U_k = 0 \tag{49}$$

in the following way (we supposed that the conformal factor $f$ and the pre-potential $W$ are given) :

- One has to solve the equations (48) or (49) for $F^S_{1,1,1}, F^S_{1,1,2}$ and $F^S_{1,2,2}$.
- Then, one can solve the equations (47) for last undefined $F^S_{2,2,2}$
- As the final step, one has to fix all coefficients by checking the equations (46).

Consider as the first example the general *Log* prepotential

$$W = \sum a_k \log(\alpha_k x_1 + \beta_k x_2). \tag{50}$$

The supersymmetry fixes the coefficients in (50) as follows:

$$\sum a_k = 0. \tag{51}$$

In the simplest case $\alpha_1 = 1$, $\beta_1 = 0$, $\alpha_2 = 0$, $\beta_2 = 1$, $a_2 = -a_1 = a$ the tensor $F^S_{ijk}$ reads

$$F^S_{111} = \frac{1 - 6x_1^2 + x_1^4 + 4x_1^2x_2^2 - x_2^4}{x_1\left(1 - x^2\right)},\ F^S_{112} = -2x_2\frac{1 + x_1^2 - x_2^2}{1 - x^2},$$
$$F^S_{122} = -2x_1\frac{1 - x_1^2 + x_2^2}{1 - x^2},\ F^S_{222} = \frac{1 - 6x_2^2 + x_2^4 + 4x_1^2x_2^2 - x_1^4}{x_2\left(1 - x^2\right)}. \tag{52}$$

The second example is the non-Calogero type solution

$$W_i = f\left(\frac{a_i}{x_i} - \frac{2(a_1+a_2)}{x^2(1-x^2)}x_i\right), \quad W_iW_i = f^2\left(\frac{a_1^2}{x_1^2} + \frac{a_2^2}{x_2^2} + 4\frac{(a_1+a_2)^2}{(1-x^2)^2}\right). \tag{53}$$

In this case, the tensor $F^S_{ijk}$ can be written as

$$\begin{aligned} F^S_{111} &= \frac{1+x^2}{x_1} + 2x_1\frac{2-x^2}{x^2} - \frac{4x_1^3}{(1-x^2)(x^2)^2} - 4\frac{a_2x_1x_2^2(1+x^2)}{x^2(a_2x_1^2 - a_1x_2^2)}, \\ F^S_{222} &= \frac{1+x^2}{x_2} + 2x_2\frac{2-x^2}{x^2} - \frac{4x_2^3}{(1-x^2)(x^2)^2} + \frac{4a_1x_1^2x_2(1+x^2)}{x^2(a_2x_1^2 - a_1x_2^2)}, \end{aligned} \tag{54}$$

$$\begin{aligned} F^S_{112} &= -2x_2 + 4\frac{(a_1+a_2)x_1^2x_2^3 - a_2x_1^2x_2(x^2)^3}{(1-x^2)(x^2)^2(a_2x_1^2 - a_1x_2^2)}, \\ F^S_{122} &= -2x_1 - 4\frac{(a_1+a_2)x_1^3x_2^2 - a_1x_1x_2^2(x^2)^3}{(1-x^2)(x^2)^2(a_2x_1^2 - a_1x_2^2)}. \end{aligned} \tag{55}$$

In this talk we analyzed the $N = 4$ supersymmetric mechanics on the conformally flat manifolds.

We started with the special Ansatz for the supercharges and found two sets of differential equations (for both cases with and without bosonic spinors) for corresponding functions entering in these Ansatz.

We found that one of the equation, rewritten in terms of differential objects, is the generalization of the WDVV to the non-flat manifolds.

For the both considered cases, in which the bosonic spinor variables were included or not, we gave the full Hamiltonian forming with the corresponding supercharges the $N = 4$ superalgebra.

Finally, we discussed some examples of the two-particles model on the two-dimensional sphere.

As further studying of this direction, it would be interesting to obtain the description of considered models within the superfiled approach.

In order to do this it is necessary to find both the corresponding superfields constrains and the Lagrangians.

We also plan to include in the final version of this work many other examples that may be of interest, in particular in their connection with the models of particles moving in the background of the extremal black holes.

Finally, it is important to analyze more general case of $N = 4$ supersymmetric mechanics both in the Hamiltonian approach and on the superfield level, when the metric function is an arbitrary one.

We are planning to attach our attention to these problems in the near future.