Partition function of free conformal fields in 3-plet representation

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AdS_{d+1}/CFT_d

free boundary CFT_d

(i) "vectorial": Φ in fundamental of $U(N)$ or $O(N)$

(ii) "adjoint": Φ in adjoint of $U(N)$ or $O(N)$

vectorial: bilinear "single-trace" operators $\Phi_i^* \partial ... \partial \Phi_i$

adjoint: multilinear single-trace operators $\mathrm{tr}(\Phi \partial ... \partial \Phi \partial ... \partial \Phi ... \partial \Phi)$

any $d = 3, 4, ...$ and any free conformal field Φ is ok

• adjoint AdS/CFT: e.g. $\lambda = g_{\mathrm{YM}}^2 N = 0$ limit of AdS_5/CFT_4
should be related to tensionless limit of string theory in AdS

• dual higher spin theory in AdS:
contains infinite set of (massless and massive) HS fields in AdS
dual to primary operators in boundary CFT

Higher representations of internal symmetry ?

• CFT field belonging e.g. to 3-fundamental ("3-plet")
general or symmetric or antisymmetric 3-index tensor

• many more irreducible singlet operators:
instead of 1d ("circle") single trace $\Phi^{ij}\partial...\partial\Phi^{jk}\partial...\partial\Phi^{kn}...\partial\Phi^{ni}$
"spatial" contractions, e.g. "tetrahedron" or "pyramid" like

$$\Phi^{ijk}\partial...\partial\Phi^{kmn}\partial...\partial\Phi^{njp}...\partial\Phi^{pmi}, \quad \text{etc}$$

[tensor theories, "triangulation" of 3-spaces]

• spectrum of "single-trace" operators with more than two fields
dual to massive fields in AdS – more intricate than in adjoint
"tensionless membrane" interpretation?

• coefficient in front of dual AdS field theory action will be N^3
(to match large N scaling of 3-point correlation functions)
cf. $AdS_7 \times S^7$ for M5-brane

Finite T singlet partition function Z:

encodes spectrum of "single-trace" ops in small T expansion

- vector case: singlet large N Z for $U(N)$ [Shenker, Yin 11] and $O(N)$ [Giombi, Klebanov, AT 14; Jevicki et al 14] matched to massless HS partition functions in AdS
- adjoint case: [Sundborg 99; Polyakov 01; Aharony et al 03] matching Z to AdS partition function [Bae, Joung, Lal 16]
- phase transition at larger $T_c \sim N^\gamma \gg 1$ in vector and $T \sim 1$ in adjoint case dual AdS interpretation (finite-size black hole) in adjoint case

- aim: compute singlet Z for free CFT in a 3-plet rep; analyse its small T expansion and match to direct operator count; large N matrix model $\to$ phase transition at $T = \frac{a}{\log N} \to 0$

Heuristic motivation: (2,0) tensor multiplet as M5-brane theory

• single M5-brane: 11d sugra solution –
free 6d CFT – $(2,0)$ tensor multiplet as w-volume theory:
selfdual $H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}$, 5 ϕ_r and 2 Weyl ψ_a

• analogy with multiple D-branes connected by open strings
[low energy – SYM – N^2 vector multiplets at weak coupling
matching leading N^2 scaling in dual supergravity]
need N^3 (2,0) multiplets to match N^3 scaling in 11d sugra

• conjecture: N^3 scaling of observables of multiple M5-branes
explained in terms of M2-branes ending on 3 M5-branes:
triple M5-brane connections by "pants-like" membrane surfaces
provide dominant contribution [Klebanov, AT 96]
leading to 3-index world volume fields

• 6d superconformal theory of multiple M5-branes?

$(2,0)$ tensor multiplets in 3-tensor rep of $SU(N)$ or $SO(N)$
[Bastianelli, Frolov, AT 99]
$(B^{ijk}_{\mu\nu}, \phi^{ijk}_r, \psi^{ijk}_a)$, $\quad i,j,k = 1,2,...,N$: $\;$ dim $\propto N^3$
• alternatively, interacting (2,0) tensor multiplets as low-E limit of tensionless 6d string: closed strings carrying 3-plet indices from virtual membranes connecting 3 parallel M5-branes
[cf. $H^{ijk}_{\mu\nu\lambda} = dB^{ijk}_{\mu\nu}$ and $F^{ij}_{\mu\nu}$ in open string (adjoint) case]
• many open questions: interacting $L = H^{ijk}_{\mu\nu\lambda} + ...$ – conf inv? only at quantum level – interacting fixed point?
• existence of well-defined large N limit ?
analogy with tensor models [Klebanov, Tarnapolsky 16]
connection with $d = 1$ SYK model [Gurau; Witten 16]
3-tensor models with distinguishable indices: large N limit described by iterated ”melonic” graphs [Gurau]

• multiple M5-brane theory should admit large N expansion
– suggested by 11d M-theory corrections to its anomalies
[Harvey et al 98; AT 00; Beccaria, AT 15]
and its free energy [Gubser, Klebanov, AT 98]

• first consider just free 3-plet tensor multiplet CFT
regardless its connection to M5-brane – should have AdS_7 dual
• More generally: study 3-plet version of AdS/CFT
– for any free CFT_d in 3-plet representation

• Aim: study singlet large N partition function
in free scalar 3-plet CFT (scalar case is general enough)
• describe spectrum of "single-trace" operators in 3-plet case,
growth with N, possible phase transitions, etc.
• compare to vector and adjoint rep cases

Partition function with singlet constraint

- free complex scalar CFT: $L = \int d^d x \, \partial^m \Phi^*_{ijk} \partial_m \Phi_{ijk}$
- singlet constraint may be implemented by coupling to flat $U(N)$ connection and integrating over its non-trivial holonomy on S^1 – over constant $U \in U(N)$
- for rep R singlet part function $Z(x)$, $x = e^{-\beta}$ given by matrix U integral with "action" depending on $\chi_R(U)$ and one-particle partition function $z_\Phi(x)$
- singlet partition function of CFT on $R \times S^{d-1}$

$$Z = \sum_{\text{singlets}} x^E, \qquad x = e^{-\beta}, \qquad \beta = 1/T$$

E of states on S^{d-1} = dimensions Δ of operators in $\mathbb{R}^d$

• single-particle partition function

$$z_\Phi(x) = \sum_i x^{E_i}$$

counts states created by Φ and its descendants mod e.o.m.
– character of corresponding rep of conf group
• for scalar, Weyl fermion, 4d vector, 6d self-dual tensor

$$z_{\mathrm{S},d}(x) = \frac{x^{\frac{d}{2}-1}(1+x)}{(1-x)^{d-1}}\,, \qquad z_{\mathrm{F},d}(x) = \frac{2^{\frac{d}{2}}\,x^{\frac{d-1}{2}}}{(1-x)^{d-1}}$$

$$z_{\mathrm{V},4}(x) = \frac{6\,x^2 - 2\,x^3}{(1-x)^3}\,, \qquad z_{\mathrm{T},6}(x) = \frac{10\,x^3 - 5\,x^4 + x^5}{(1-x)^5}$$

• for single boson Φ in a real rep R of $U(N)$ (e.g. $R = N \oplus \overline{N}$)

$$Z = \sum_{n_1 \geq 0} x^{n_1 E_1} \sum_{n_2 \geq 0} x^{n_2 E_2} \text{...\# singlets } \mathrm{sym}^{n_1}(R) \otimes \mathrm{sym}^{n_2}(R)\text{...}$$

- singlet constraint: by integrating over the symmetry group

$$Z = \int dU \prod_i \sum_{n_i \geq 0} x^{n_i E_i} \chi_{\mathrm{sym}^{n_i}(R_i)}(U)$$

- using explicit form of χ of $\mathrm{sym}^n(R)$ [Skagerstam 83]

$$Z = \int dU \, \exp\left\{ \sum_{m=1}^{\infty} \frac{1}{m} z_\Phi(x^m) \, \chi_R(U^m) \right\}, \qquad z_\Phi(x) = \sum_i x^{E_i}$$

in boson + fermion case $z(x^m) \;\to\; z_B(x^m) + (-1)^{m+1} z_F(x^m)$

- Examples of χ_R:

$$\text{vector}: \quad N \oplus \overline{N} \qquad\qquad \chi_R = \mathrm{tr}(U) + \mathrm{tr}(U^\dagger)$$

$$\text{adjoint}: \qquad\qquad \chi_R = \mathrm{tr}(U)\,\mathrm{tr}(U^\dagger)$$

$$\text{3-plet}: \quad N^{\otimes 3} \oplus \overline{N}^{\otimes 3} \qquad\qquad \chi_R = [\mathrm{tr}(U)]^3 + [\mathrm{tr}(U^\dagger)]^3$$

p-plet: product of p fundamentals: $\quad R = N^{\otimes p} \oplus \overline{N}^{\otimes p}$

$$\chi_{N^{\otimes p} \oplus \overline{N}^{\otimes p}}(U) = \left[\mathrm{tr}(U)\right]^p + \left[\mathrm{tr}(U^\dagger)\right]^p$$

(anti) symmetric product

$$\chi_{(N \otimes N \otimes N)_{\text{(anti)sym}}}(U) = \tfrac{1}{6}\left[\chi_N(U)\right]^3 \pm \tfrac{1}{2}\,\chi_N(U)\,\chi_N(U^2) + \tfrac{1}{3}\,\chi_N(U^3)$$

Derivation from scalar partition function on $S^1_\beta \times S^{d-1}$:

singlet projection implemented by coupling Φ to a $A_\mu = U^{-1}\partial_\mu U$
constant part of A_0 cannot be gauged away
e.g. in vector case: complex $U(N)$ scalars Φ_i $\quad (t \in (0,\beta))$

$$\partial_t^2 \to (\partial_t + A_0)^2, \quad A_0 = U^{-1}\partial_0 U, \quad U(t) = \mathrm{diag}(e^{i\frac{\alpha_1}{\beta}t}, ..., e^{i\frac{\alpha_N}{\beta}t})$$

$$Z = \int \prod_{k=1}^{N} d\alpha_k \ e^{-\tilde{F}(\alpha,\beta)}\,, \qquad \tilde{F} = -\sum_{i\neq j}^{N} \ln|\sin\tfrac{\alpha_i-\alpha_j}{2}| + \bar{F}(\alpha,\beta)$$

$$\bar{F} = \ln\det\left[-(\partial_t + A_0)^2 + \Delta_{S^{d-1}}\right] = \sum_{i=1}^{N}\sum_{k,n}^{\infty} \mathrm{d}_n \ln\left[\frac{(2\pi k+\alpha_i)^2}{\beta^2} + \omega_n^2\right]$$

$$\bar{F} = -\sum_{m=1}^{\infty} \tfrac{1}{m}\, b_m(\alpha)\, z_\Phi(m\beta)\,, \qquad b_m(\alpha) = \chi_R(U^m(\beta)) = 2\sum_{i=1}^{N} \cos m\alpha_i$$

$N = \infty$ limit of low T expansion of singlet Z

- expand U integral in powers of $x = e^{-\beta}$, then take $N \to \infty$
- vector and adjoint: low T, $N = \infty$ expansion is convergent
- 3-plet case: expansion is only asymptotic ($x_c = 0$)
- reason – rapid growth of # of singlets with dimension:

phase transition at small $T_c \sim (\log N)^{-1} \to 0$
i.e. low T phase effectively shrinks to $T = 0$ for $N = \infty$

- $N = \infty$ limit: counting of singlet states simplifies

Z expressed in terms of the "single-trace" $Z_{\text{s.t.}}(x)$=
counting only fully-connected (indecomp.) contractions

$$\log Z(x) \equiv \sum_{m=1}^{\infty} \frac{1}{m} Z_{\text{s.t.}}(x^m)$$

Vector and adjoint representation cases

• vector case: singlets in $\mathrm{sym}^n(N \oplus \overline{N})$ products of bilinears

e.g. bilinear singlets $\sum_{ss'} c_{ss'} \sum_i \partial^s \overline{\Phi}_i \partial^{s'} \Phi_i$.

"single-trace" partition function is square of single-particle one

$$Z^{\text{vec}}_{\text{s.t.}}(x) = \left[z_\Phi(x)\right]^2$$

all singlets – $N = \infty$ singlet partition function

$$\log Z^{\text{vec}} = \sum_{m=1}^{\infty} \frac{1}{m} \left[z_\Phi(x^m)\right]^2$$

• adjoint case: singlets as products of single-trace operators

Z for single-trace ops from Polya enumeration theorem

[Sundborg 99; Polyakov 01]

$$Z^{\text{adj}}_{\text{s.t.}} = -\sum_{m=1}^{\infty} \frac{\varphi(m)}{m} \log\left[1 - z_\Phi(x^m)\right]$$

$\varphi(m)$ – Euler's totient function counting positive integers up to a given integer m that are relatively prime to m

• $N = \infty$ singlet partition function – all multi-trace singlets

$$\log Z^{\rm adj} = \sum_{m=1}^{\infty} \frac{1}{m} Z^{\rm adj}_{\rm s.t.}(x^m) = - \sum_{m=1}^{\infty} \log\left[1 - z_\Phi(x^m)\right]$$

AdS/CFT perspective:

• vector case: bilinear primaries – massless HS in AdS total partition function matches 1-loop AdS partition function [Shenker, Yin 11; Giombi, Klebanov, AT 14; Beccaria, AT 14]

• adjoint case: single traces – towers of massless and massive HS in AdS; on group-theoretic basis expect to match multi-particle Z with its AdS counterpart [Bae, Joung, Lal 16]

Low temperature expansion of Z and counting of operators

expansion of Z in $x = e^{-\beta}$ encodes counting of singlets

$$Z = \int dU \, \exp\left\{ \sum_{m=1}^{\infty} \frac{1}{m} z_{\Phi}(x^m) \, \chi_R(U^m) \right\}, \qquad z_{\Phi}(x) = \sum_i x^{E_i}$$

$$I(\boldsymbol{a}, \boldsymbol{b}) = \int dU \, \textstyle\prod_{\ell \geq 1} (\mathrm{tr}\, U^{\ell})^{a_\ell} \, \overline{(\mathrm{tr}\, U^{\ell})^{b_\ell}} \to_{N\to\infty} \textstyle\prod_{\ell\geq 1} \ell^{a_\ell} \, a_\ell! \, \delta_{a_\ell, b_\ell}$$

- vector case: if Φ is 4d scalar with $z_\Phi(x) = z_{S,4}(x) = \frac{x((1+x)}{(1-x)^3}$

$$Z^{\mathrm{vec}}_{S,4} = 1 + x^2 + 8\,x^3 + 35\,x^4 + 112\,x^5 + 330\,x^6 + 944\,x^7 + \ldots$$

$N \to \infty$ and $x \to 0$ commute; ∞ conv. radius: $T_c \sim N^{\gamma} \to \infty$

- adjoint case: more operators at higher dimensions

$$Z^{\mathrm{adj}}_{S,4} = 1 + x + 6\,x^2 + 20\,x^3 + 75\,x^4 + 252\,x^5 + 914\,x^6 + 3160\,x^7 + \ldots$$

finite radius of convergence: $T_c \sim N^0 \sim 1$

Comparison to direct counting of operators:

(i) vector case: "single-trace" partition function

4d scalar $z_{S,4}(x) = \frac{x(1+x)}{(1-x)^3}$, $\quad [z_{S,4}(x)]^2 = x^2 + 8\,x^3 + \ldots$:

dim 2: one operator $\overline{\varphi}_i \varphi_i$

dim 4: $4+4$ operators $\overline{\varphi}_i\, \partial_\mu\, \varphi_i$ and $\partial_\mu \overline{\varphi}_i\, \varphi_i$

(ii) adjoint case: single-trace $Z^{\rm adj}_{\rm s.t.} = x + 5\,x^2 + \ldots$:

dim 1: one operator $\mathrm{tr}(\varphi)$

dim 2: $1+4=5$ operators $\mathrm{tr}(\varphi^2)$ and $\partial_\mu\, \mathrm{tr}(\varphi)$

(iii) 3-plet representation:

large N limit of small x expansion of Z for 4d scalar

$$Z^{3-\text{plet}}_{S,4} = 1 + 6\,x^2 + 48\,x^3 + 396\,x^4 + 3504\,x^5 + 35580\,x^6$$

$$+\,381216\,x^7 + 4408956\,x^8 + 53647632\,x^9 + 689785308\,x^{10} + \ldots$$

symmetric $(+)$ or antisymmetric $(-)$ 3-index reps

$$Z_{\mathrm{S},4}^{3-\mathrm{plet}^+} = 1 + x^2 + 8\,x^3 + 36\,x^4 + 120\,x^5 + 404\,x^6 + 1368\,x^7 + \ldots$$

$$Z_{\mathrm{S},4}^{3-\mathrm{plet}^-} = 1 + x^2 + 8\,x^3 + 36\,x^4 + 120\,x^5 + 403\,x^6 + 1360\,x^7 + \ldots$$

fewer operators as some contractions become equivalent

compare to direct counting of operators:

• dim 2: singlets built out of scalar $\Phi = (\varphi_{ijk})$

$(\overline{\varphi}\,\varphi) = \overline{\varphi}_{ijk}\varphi_{i'j'k'}$, $i'j'k'$ = permutation of ijk: $3! = 6$ different

• dim 3: singlets $(\overline{\varphi}\,\partial_\mu\varphi)$, $(\partial_\mu\overline{\varphi}\,\varphi)$, $2 \times 4 \times 6 = 48$ operators

• dim 4: bilinears: $(\overline{\varphi}\,\partial_\mu\partial_\nu\varphi)$, $(\partial_\mu\partial_\nu\overline{\varphi}\,\varphi)$, $(\partial_\mu\overline{\varphi}\partial_\nu\varphi)$

ignoring $\sim \partial^\mu\partial_\mu\varphi = 0$ $(9 \times 2 + 4 \times 4) \times 6 = 204$ operators

quartic: (i) reducible contraction $(\overline{\varphi}\varphi)(\overline{\varphi}\varphi)$: $\frac{1}{2} \times 6 \times 7 = 21$

(ii) irreducible "single-trace" $(\overline{\varphi}\varphi\overline{\varphi}\varphi)$, $\overline{\varphi}_{ijk}\varphi_{ijl} = X_{kl}$

$3^2 \times 2 = 18$ X, contracting $\frac{1}{2} \times 18 \times 19 = 171$

dim 4 singlets: $204 + 21 + 171 = 396$ in agreement with x^4 term

• similar results in 6d, for fields of tensor multiplet, etc.

Comparing 3-plet case to adjoint case:

• number of singlets grows much faster with dim of operator implies non-convergence of small x expansion of Z

• analog of "Hagedorn" transition in adjoint case happens at much lower $T_c \sim (\log N)^{-1} \to 0$ at $N \to \infty$

Closed expression for low T expansion of Z at $N \to \infty$

$$Z = \prod_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{z_\Phi(x^m)}{m} \right)^k \int dU \left[\chi_R(U^m) \right]^k$$

for p-plet $N^{\otimes p}$ rep of $U(N)$: $\quad \chi_R(U) = \left[\mathrm{tr}(U) \right]^p + \left[\mathrm{tr}(U^\dagger) \right]^p$

$$Z^{p\text{-plet}} = \prod_{m=1}^{\infty} F_p\left(m^{p-2} \left[z_\Phi(x^m) \right]^2 \right)$$

$F_p(y) \equiv \sum_{k=0}^{\infty} b_k\, y^k$, $\qquad b_k = \frac{(p\,k)!}{(k!)^2}$, $\qquad p = 1, 2, 3, \ldots$

e.g. for $p = 1$ and $p = 2$: $\quad F_1(y) = e^y$, $\quad F_2(y) = \frac{1}{\sqrt{1-4y}}$

$$\log Z^{1\text{-plet}} = \sum_{m=1}^{\infty} \frac{1}{m} \left[z_\Phi(x^m) \right]^2,$$

$$\log Z^{2\text{-plet}} = -\tfrac{1}{2} \sum_m \log\left(1 - 4 \left[z_\Phi(x^m) \right]^2 \right)$$

$$\log Z^{\mathrm{adj}} = -\sum_{m=1}^{\infty} \log\left[1 - z_{\Phi}(x^m)\right]$$

• series F_p no longer converges starting with $p = 3$

for $p \geq 3$ get only formal generating function for the spectrum

• $p = 3$: "resum" the series by replacing $(3\,k)!$ in b_k by $\int_0^\infty dt\, e^{-t}\, t^{3\,k}$

$$F_3(y) \;\to\; \widetilde{F}_3(y) = \tfrac{1}{6}\sqrt{\tfrac{\pi}{3}\, y^{-1}}\; e^{-\frac{1}{54}\, y^{-1}} \left[I_{\frac{1}{6}}\left(\tfrac{1}{54}\, y^{-1}\right) + I_{-\frac{1}{6}}\left(\tfrac{1}{54}\, y^{-1}\right)\right]$$

$\widetilde{F}_3(y)$ has a branch cut on negative real axis, smooth for $y \geq 0$

power series defining $F_3(y)$ is asymptotic expansion of $\widetilde{F}_3(y)$

alternative: Borel resummation of $F_3(y)$

$$\widetilde{F}_3^B(y) = \int_0^\infty dt\, e^{-t} \sum_{k=0}^{\infty} \frac{b_k}{k!}\,(yt)^k = \tfrac{1}{3}\sqrt{-\tfrac{1}{3\pi}\, y^{-1}}\; e^{-\frac{1}{54}\, y^{-1}}\; K_{\frac{1}{6}}\left(-\tfrac{1}{54}\, y^{-1}\right)$$

• e.g. for 4d scalar in 3-plet representation

$$Z_{S,4}^{3-\text{plet}} = \prod_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{(3\,k)!}{(k!)^2}\, m^k \left[z_\Phi(x^m)\right]^{2k}, \qquad z_\Phi(x) = \frac{x(1+x)}{(1-x)^3}$$

• same expression for other fields with corresponding z_Φ
encodes number of singlet operators built out of Φ in 3-plet rep

• similar expression for p-tensor with distinguished indices
transforming under separate $U(N)$'s:
singlet Z found by gauging the full $[U(N)]^p$ group;
less singlet operators but again large N limit of small x
expansion of Z becomes only asymptotic starting with $p = 3$

Large N partition function and phase transitions

rapid growth of # of states with dim of $U(N)$ rep

recall adjoint case: Z diverges when $z_\Phi(x) = 1 \to x_c = e^{-\beta_c}$

$$\log Z^{\text{adj}} = \sum_{m=1}^{\infty} \frac{1}{m} Z^{\text{adj}}_{\text{s.t.}}(x^m) = -\sum_{m=1}^{\infty} \log\left[1 - z_\Phi(x^m)\right]$$

well defined for $\beta > \beta_c$, diverges $Z \sim (\beta - \beta_c)^{-1}$ for $\beta \to \beta_c$

cf. Hagedorn behaviour $\rho(E) \simeq e^{\beta_c E}$, $\quad Z = \int dE\, \rho(E)\, x^E$

• higher T: find dominant distribution of eigenvalues of U

$N \to \infty$ distribution approximated by density $\rho(\alpha)$, $\alpha \in (-\pi, \pi)$

$$\rho(\alpha) \geq 0, \qquad \int_{-\pi}^{\pi} d\alpha\, \rho(\alpha) = 1.$$

• transition from phase where $\rho > 0$ on $(-\pi, \pi)$

to phase where $\rho > 0$ only on $(-\alpha_0, -\alpha_0) \subset (-\pi, \pi)$

- transition: balance measure term $\sim N^2$ and character term

$$N^2 \sim N^p \, z_\Phi(x_c), \qquad\qquad x_c = e^{-1/T_c}$$

$p = 1, 2, 3, ...$ for the vector, adjoint, 3-plet representation, etc.

- vector case: $x_c \to 1$ as $N \to \infty$ and since $z_\Phi(x) \overset{x\to 1}{\sim} T^{d-1}$

$$T_c^{\text{vec}} \sim N^{\frac{1}{d-1}} \to \infty$$

- adjoint case: T_c is independent of N

$$T_c^{\text{adj}} \sim 1$$

- 3-plet case: T_c vanishes as $N \to \infty$, e.g. for a scalar

$$T_c^{3-\text{plet}} \sim \tfrac{d-2}{2} \frac{1}{\log N} \to 0$$

similar for other fields, e.g. 6d tensor multiplet $T_c^{3-\text{plet}} \sim \frac{10}{\log N}$.

• Summary: at large N 1st order discontinuous transition between "low T" phase with $\rho > 0$ everywhere on $(-\pi, \pi)$ and "high T" phase with $\rho > 0$ only for $|\alpha| \leq \alpha_0$ with $\alpha_0 \sim (N z_\Phi)^{-1/2}$; transition point at $(N z_\Phi)_c = \frac{9}{16}$ for any T for sufficiently large N get "high T" phase

3-plet case – large N: $T_c \to 0$: low T phase is shrinking

Large N limit in terms of eigenvalue density

integration over U in terms of eigenvalues $\{e^{i\,\alpha_i}\}$ $\quad (-\pi < \alpha_i \leq \pi)$

$$Z = \int d\boldsymbol{\alpha}\, e^{-S(\boldsymbol{\alpha},x)}\,, \qquad \int dU = \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\alpha_i \prod_{i<j} \sin^2 \tfrac{\alpha_i - \alpha_j}{2}$$

$$S(\boldsymbol{\alpha}, x) = -\tfrac{1}{2} \sum_{i \neq j} \log \sin^2 \tfrac{\alpha_i - \alpha_j}{2} + \sum_{m=1}^{\infty} c_m(x)\, \mathcal{V}(m\, \boldsymbol{\alpha})$$

$$c_m \equiv -\tfrac{1}{m}\, z_\Phi(x^m), \qquad \mathcal{V}^{\text{vec}}(\boldsymbol{\alpha}) = 2 \textstyle\sum_i^N \cos \alpha_i$$

$$\mathcal{V}^{\text{adj}}(\boldsymbol{\alpha}) = \sum_{i,j}^{N} \cos(\alpha_i - \alpha_j)$$

$$\mathcal{V}^{3-\text{plet}}(\boldsymbol{\alpha}) = 2 \sum_{i,j,k}^{N} \cos(\alpha_i + \alpha_j + \alpha_k)$$

integration over $\boldsymbol{\alpha} \to \ \rho(\alpha)$ periodic on $\alpha \in (-\pi, \pi)$

$$S(\rho, x) = S_M(\rho) + V(\rho, x)\ , \qquad\qquad \rho(\alpha) = \frac{1}{N}\ \sum_{n=1}^{N} \delta(\alpha - \alpha_i)$$

$$S_M = N^2 \int d\alpha\, d\alpha'\, K(\alpha - \alpha')\, \rho(\alpha)\, \rho(\alpha'),$$

$$K(\alpha) = -\tfrac{1}{2}\ \log(2 - 2\,\cos\alpha) = \sum_{m=1}^{\infty} \tfrac{1}{m} \cos(m\alpha)$$

$$V^{\rm vec} = 2\,N \int d\alpha\, \rho(\alpha)\ \sum_{m=1}^{\infty} c_m(x)\ \cos(m\,\alpha)$$

$$V^{\rm adj} = N^2 \int d\alpha\, d\alpha' \rho(\alpha)\, \rho(\alpha')\ \sum_{m=1}^{\infty} c_m(x)\ \cos\left[m\,(\alpha - \alpha')\right]$$

$$V^{3-\rm pl} = 2N^3 \int d\alpha d\alpha' d\alpha'' \rho(\alpha)\rho(\alpha')\rho(\alpha'')\ \sum_{m=1}^{\infty} c_m(x) \cos\left[m(\alpha + \alpha' + \alpha'')\right]$$

Vector and adjoint cases

expand $\rho(\alpha)$ in Fourier modes

$$\rho(\alpha) = \frac{1}{2\,\pi} + \frac{1}{N}\Big[\frac{1}{\pi}\,\sum_{m=1}^{\infty} \rho_m^+ \cos(m\alpha) + \frac{1}{\pi}\,\sum_{m=1}^{\infty} \rho_m^- \sin(m\alpha)\Big]$$

$$S_M = \sum_{m=1}^{\infty} \frac{1}{m}\,\left[(\rho_m^+)^2 + (\rho_m^-)^2\right]$$

$$V^{\text{vec}} = 2\,\sum_{m=1}^{\infty} c_m\,\rho_m^+\,, \qquad V^{\text{adj}} = \sum_{m=1}^{\infty} c_m\,\left[(\rho_m^+)^2 + (\rho_m^-)^2\right]$$

- vector case: action is stationary at

$$\rho_m^+ = -\,m\,c_m = z_\Phi(x^m)\,, \qquad \rho_m^- = 0$$

get same expression for Z as by Gaussian integral over $\rho_m^{\pm}$

$$\log Z^{\mathrm{vec}} = \sum_{m=1}^{\infty} \frac{1}{m} \left[z_\Phi(x^m) \right]^2$$

- adjoint case: integrating over $\rho_m^{\pm}$

$$S^{\mathrm{adj}} = \sum_{m=1}^{\infty} \frac{1 - z_\Phi(x^m)}{m} \left[(\rho_m^+)^2 + (\rho_m^-)^2 \right]$$

$$\log Z^{\mathrm{adj}} = - \sum_{m=1}^{\infty} \log \left[1 - z_\Phi(x^m) \right]$$

- small T: small x and $z_\Phi(x)$ near $\rho =$const expansion is ok
- larger T: transition where ρ is zero only on $(-\alpha_0, \alpha_0) \subset (-\pi, \pi)$

  vector case: transition at $T_c \sim N^{\frac{1}{d-1}} \gg 1$

  adjoint case: critical T from condition $z_\Phi(x) = 1 \quad \rightarrow \quad T_c \sim 1$

• low T phase: action and Z not depend on $N \gg 1$: $\log Z \sim N^0$

• $T > T_c$ phase: stationary point solution for $\rho(\alpha)$
due to a balance between measure and potential:
action at stationary point scales as measure term $\sim N^2$
i.e. in high T phase: $\log Z \sim N^2$

3-plet case

$$S = \sum_{m=1}^{\infty} \tfrac{1}{m} \left[(\rho_m^+)^2 + (\rho_m^-)^2 \right] + V^{3-\text{plet}}(\rho^\pm, x)$$

$$V^{3-\text{plet}} = -2 \sum_{m=1}^{\infty} \tfrac{1}{m} z_\Phi(x^m) \left[(\rho_m^+)^3 - 3\, \rho_m^+ \, (\rho_m^-)^2 \right]$$

• action unbounded from below – integral over $\rho_m^\pm$ diverges

• $\rho = \frac{1}{2\pi}$=const is saddle not minimum even at low T

• $\rho_m^\pm$ may be large – violating positivity of $\rho(\alpha)$

• phase transition: ρ^3 potential becomes of order ρ^2 measure condition: $N^2 \sim N^p z_\Phi(x), \quad p = 1, 2, 3$

• vector case: measure term $\sim N^2$ against potential $\sim N z_\Phi(x)$

• adjoint case: both terms are of the same order $\sim N^2$

• 3-plet case: potential scales as $N^3 z_\Phi(x)$ and get $N^2 \sim N^3 z_\Phi(x)$

• here low T phase shrinking with increasing N: $\quad T_c \to 0$

action and $\log Z$ scaling as N^2 at stationary-point solution

Solution for eigenvalue density at large N

stationary point condition in terms of $\rho(\alpha)$:

$$\int d\alpha' \rho(\alpha') \cot \tfrac{\alpha-\alpha'}{2}$$

$$= 6N \sum_{m=1}^{\infty} z_{\Phi}(x^m) \int d\alpha' d\alpha'' \rho(\alpha')\rho(\alpha'') \sin\left[m(\alpha+\alpha'+\alpha'')\right]$$

assume ρ is symmetric and supported on $(-\alpha_0, \alpha_0)$

$$\int d\alpha' \, \rho(\alpha') \, \cot \tfrac{\alpha-\alpha'}{2} = 2 \sum_{m=1}^{\infty} a_m \, \rho_m^2 \, \sin(m\,\alpha),$$

$a_m = 3\,N\,z_{\Phi}(x^m),$ $\rho_m = \int d\alpha \, \rho(\alpha) \, \cos(m\,\alpha)$

$\rho(\alpha) = \frac{1}{\pi} \sqrt{\sin^2 \frac{\alpha_0}{2} - \sin^2 \frac{\alpha}{2}} \, \sum_{k=1}^{\infty} Q_k \, \cos\left[(k-\tfrac{1}{2})\,\alpha\right]$

$Q_k = 2 \sum_{\ell=0}^{\infty} a_{k+\ell} \, \rho_{k+\ell}^2 \, P_{\ell}(\cos \alpha_0)$

• model with just one harmonic ρ_1: good for large β

when $x = e^{-\beta} \ll 1$ and a_m decreases with m

$$u \equiv \sin^2 \tfrac{\alpha_0}{2} \,, \qquad \alpha_0 = \left[\tfrac{3}{2} N \, z_\Phi(e^{-\beta})\right]^{-1/2} + ...$$

• for each temperature and N such that $N \, z_\Phi(e^{-\beta}) > \frac{9}{16}$

$$\rho(\alpha) = \tfrac{1}{\pi \, \sin^2 \frac{\alpha_0}{2}} \sqrt{u - \sin^2 \tfrac{\alpha}{2}} \, \cos \tfrac{\alpha}{2}$$

$$|\alpha| < \alpha_0 \,; \qquad \rho(\alpha) = 0 \,, \quad \alpha_0 < |\alpha| \leq \pi$$

$$\tfrac{3}{2} \, u \, (2 - u)^2 = \left[N \, z_\Phi(e^{-\beta})\right]^{-1}$$

for $\frac{9}{16} < N \, z_\Phi(e^{-\beta}) < \frac{2}{3}$ 2 solutions $0 < u_1 < u_2 < 1$

• conclusions supported by numerical analysis

Summary

• singlet partition function Z of free CFT in higher reps: for rank ≥ 3 tensors # of singlet states/operators grows so fast with energy/dimension that small T expansion of Z has 0 radius of convergence in $N = \infty$ limit

• reflected in critical $T_c^{3-\text{plet}} \sim \frac{1}{\log N} \to 0$ at $N \to \infty$

• for large but finite N get two phases: $T < T_c$ and $T > T_c$
$F = -\log Z \sim N^2$ in high T phase (for all reps)

• similar behaviour for singlet Z of p-fundamental rep of $U(N)$ and for $[U(N)]^p$ invariant p-tensors with inequivalent indices

• AdS dual of free p-plet or p-tensor CFT ?
rich spectrum: infinite towers of massive HS fields in AdS
in addition to massless HS tower (present as in vector case)
cf. "tensionless string" spectrum in adjoint case;
"tensionless membrane" spectrum in $p = 3$ case?
• dual AdS action? inverse coupling $\sim N^3$ to match
large N correlation functions in free 3-plet CFT
• low-T phase: large N free energy $F = $ 1-loop $\log Z$
of all HS fields in thermal AdS $\sim N^0$ – to match F in CFT
• high-T phase: boundary CFT free energy $F \sim N^2$
(i) adjoint case (AdS action $\sim N^2$):
agrees with AdS black hole free energy/entropy scaling for
finite (in AdS units) size BH: $T_c \sim T_H \sim 1$ [Witten 98]
(ii) vectorial case (AdS action $\sim N$):
$T_c \sim N^\gamma \to \infty$: high T phase is not attainable

classical thermal object would give $F \sim N$ not N^2

$T_c \sim T_H \to \infty$: BH of 0 (Planck length) size [Shenker, Yin 11]

[cf. no stable AdS-Schwarzschild BH solution in HS theory]

3-plet case:

• 1-loop Z in thermal AdS for full spectrum

of AdS fields dual to singlet operators in large N limit

(i) should also be given by asymptotic series matching

low-T phase expression for boundary $\log Z \sim N^0$

(ii) in high-T phase $\log Z \sim N^2$

while possible contribution from classical AdS action $\sim N^3$

$T_c \sim T_H \sim (\log N)^{-1} \to 0$:

as if size of BH is of order of AdS scale

$N \to \infty$: no low-T phase, only high-T (opposite to vector)

• these conclusions may change in interacting 3-plet CFT ?
Examples? in 3d?
T_c may become finite at non-trivial large N fixed point ?

• $(2, 0)$ tensor multiplet theory in 6d should have
AdS_7 dual with a supergravity limit in the $N \to \infty$ limit
admitting BHs and thus predicting N^3 scaling of free energy