A finite element method for incompressible Navier-Stokes equations in a time-dependent domain

Yuri Vassilevski[1,2]
Maxim Olshanskii[3] **Alexander Lozovskiy**[1]
Alexander Danilov[1,2]

[1]Institute of Numerical Mathematics RAS
[2]Moscow Institute of Physics and Technology
[3]University of Houston

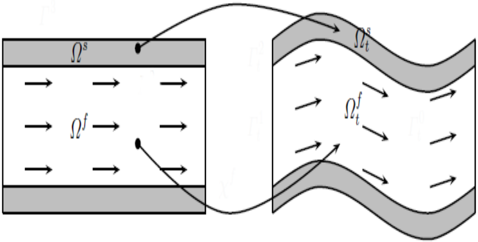
Ginzburg Centennial Conference on Physics

May 30, 2017, Moscow

The work was supported by the Russian Science Foundation

Fluid-Structure Interaction problem

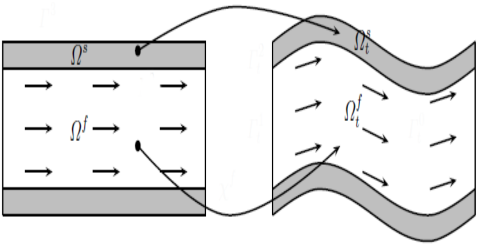
Prerequisites for FSI



- reference subdomains Ω_f, Ω_s

Fluid-Structure Interaction problem

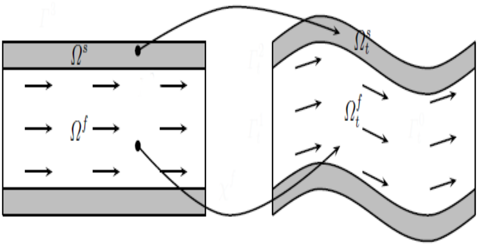
Prerequisites for FSI



- reference subdomains Ω_f, Ω_s
- transformation ξ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$

Fluid-Structure Interaction problem

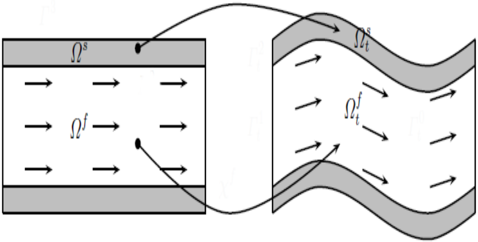
Prerequisites for FSI



- reference subdomains Ω_f, Ω_s
- transformation ξ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in $\widehat{\Omega} := \Omega_f \cup \Omega_s$

Fluid-Structure Interaction problem

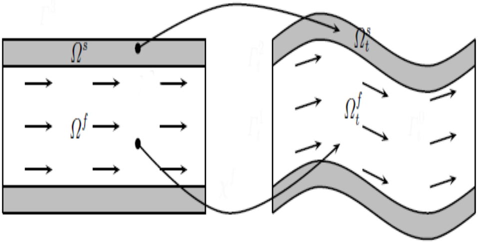
Prerequisites for FSI



- reference subdomains Ω_f, Ω_s
- transformation ξ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in $\widehat{\Omega} := \Omega_f \cup \Omega_s$
- $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$

Fluid-Structure Interaction problem

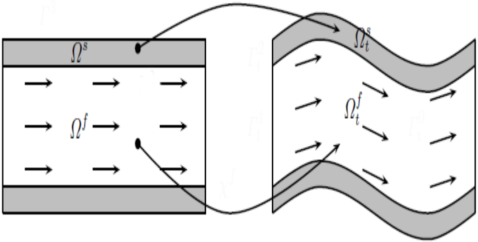
Prerequisites for FSI



- reference subdomains Ω_f, Ω_s
- transformation ξ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in $\widehat{\Omega} := \Omega_f \cup \Omega_s$
- $\xi(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\xi = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensors $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_s$

Fluid-Structure Interaction problem

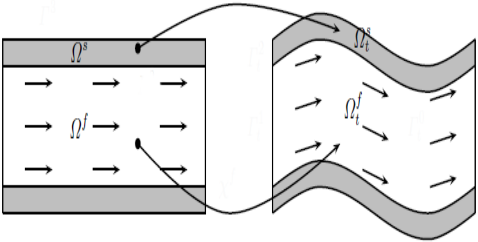
Prerequisites for FSI



- reference subdomains Ω_f, Ω_s
- transformation ξ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in $\widehat{\Omega} := \Omega_f \cup \Omega_s$
- $\xi(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\xi = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensors $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_s$
- pressures p_f,p_s

Fluid-Structure Interaction problem

Prerequisites for FSI



- reference subdomains Ω_f, Ω_s
- transformation $\boldsymbol{\xi}$ maps Ω_f, Ω_s to $\Omega_f(t)$, $\Omega_s(t)$
- $\mathbf{v}$ and $\mathbf{u}$ denote velocities and displacements in $\widehat{\Omega} := \Omega_f \cup \Omega_s$
- $\boldsymbol{\xi}(\mathbf{x}) := \mathbf{x} + \mathbf{u}(\mathbf{x})$, $\mathbf{F} := \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J := \det(\mathbf{F})$
- Cauchy stress tensors $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_s$
- pressures p_f,p_s
- densities ρ_s, ρ_f are constant

Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{cases} \rho_s^{-1}\mathrm{div}\left(J\boldsymbol{\sigma}_s\mathbf{F}^{-T}\right) & \text{in } \Omega_s, \\ (J\rho_f)^{-1}\mathrm{div}\left(J\boldsymbol{\sigma}_f\mathbf{F}^{-T}\right) - \nabla\mathbf{v}\left(\mathbf{F}^{-1}\left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\right)\right) & \text{in } \Omega_f \end{cases}$$

Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{cases} \rho_s^{-1} \mathrm{div}\left(J \boldsymbol{\sigma}_s \mathbf{F}^{-T}\right) & \text{in } \Omega_s, \\ (J\rho_f)^{-1} \mathrm{div}\left(J \boldsymbol{\sigma}_f \mathbf{F}^{-T}\right) - \nabla \mathbf{v} \left(\mathbf{F}^{-1}\left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\right)\right) & \text{in } \Omega_f \end{cases}$$

Kinematic equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \quad \text{in } \Omega_s$$

Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{cases} \rho_s^{-1} \operatorname{div}\left(J \boldsymbol{\sigma}_s \mathbf{F}^{-T}\right) & \text{in } \Omega_s, \\ (J\rho_f)^{-1} \operatorname{div}\left(J \boldsymbol{\sigma}_f \mathbf{F}^{-T}\right) - \nabla \mathbf{v} \left(\mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \right) \right) & \text{in } \Omega_f \end{cases}$$

Kinematic equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \quad \text{in } \Omega_s$$

Fluid incompressibility

$$\operatorname{div}\left(J \mathbf{F}^{-1} \mathbf{v}\right) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J \nabla \mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

Fluid-Structure Interaction problem

Universal equations in reference subdomains

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = \begin{cases} \rho_s^{-1} \mathrm{div}\left(J \boldsymbol{\sigma}_s \mathbf{F}^{-T}\right) & \text{in } \Omega_s, \\ (J\rho_f)^{-1} \mathrm{div}\left(J \boldsymbol{\sigma}_f \mathbf{F}^{-T}\right) - \nabla \mathbf{v} \left(\mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \right) \right) & \text{in } \Omega_f \end{cases}$$

Kinematic equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v} \quad \text{in } \Omega_s$$

Fluid incompressibility

$$\mathrm{div}\left(J \mathbf{F}^{-1} \mathbf{v}\right) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J \nabla \mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathbf{I} + \mu_f ((\nabla \mathbf{v}) \mathbf{F}^{-1} + \mathbf{F}^{-T} (\nabla \mathbf{v})^T) \quad \text{in } \Omega_f$$

FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathbf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathbf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

Monolithic approach: Extension of the displacement field to the fluid domain

$$\begin{aligned} G(\mathbf{u}) &= 0 \quad \text{in } \Omega_f, \\ \mathbf{u} &= \mathbf{u}^* \quad \text{on } \partial\Omega_f \end{aligned}$$

FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathbf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

Monolithic approach: Extension of the displacement field to the fluid domain

$$\begin{aligned} G(\mathbf{u}) &= 0 \quad \text{in } \Omega_f, \\ \mathbf{u} &= \mathbf{u}^* \quad \text{on } \partial\Omega_f \end{aligned}$$

for example, vector Laplace equation or elasticity equation

FSI problem

User-dependent equations in reference subdomains

Constitutive relation for the solid stress tensor

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(J, \mathbf{F}, p_s, \lambda_s, \mu_s, \dots) \quad \text{in } \Omega_s$$

Monolithic approach: Extension of the displacement field to the fluid domain

$$\begin{aligned} G(\mathbf{u}) &= 0 \quad \text{in } \Omega_f, \\ \mathbf{u} &= \mathbf{u}^* \quad \text{on } \partial\Omega_f \end{aligned}$$

for example, vector Laplace equation or elasticity equation

+ Initial, boundary, interface conditions $(\boldsymbol{\sigma}_f \mathbf{F}^{-T}\mathbf{n} = \boldsymbol{\sigma}_s \mathbf{F}^{-T}\mathbf{n})$

Numerical scheme

- Conformal triangular or tetrahedral mesh Ω_h in $\widehat{\Omega}$
- LBB-stable pairs for velocity and pressure P_2/P_1 or P_2/P_0
- Fortran open source software `Ani2D`, `Ani3D` (Advanced numerical instruments 2D/3D)
  `http://sf.net/p/ani2d/` `http://sf.net/p/ani3d/`:
  - mesh generation
  - FEM systems
  - algebraic solvers

Numerical scheme

Find $\{\mathbf{u}^{k+1}, \mathbf{v}^{k+1}, p^{k+1}\} \in \mathbb{V}_h^0 \times \mathbb{V}_h \times \mathbb{Q}_h$ s.t.

$$\mathbf{v}^{k+1} = \mathbf{g}_h(\cdot, (k+1)\Delta t) \text{ on } \Gamma_{f0}, \quad \left[\frac{\partial \mathbf{u}}{\partial t}\right]_{k+1} = \mathbf{v}^{k+1} \text{ on } \Gamma_{fs}$$

Numerical scheme

Find $\{\mathbf{u}^{k+1}, \mathbf{v}^{k+1}, p^{k+1}\} \in \mathbb{V}_h^0 \times \mathbb{V}_h \times \mathbb{Q}_h$ s.t.

$$\mathbf{v}^{k+1} = \mathbf{g}_h(\cdot, (k+1)\Delta t) \text{ on } \Gamma_{f0}, \quad \left[\frac{\partial \mathbf{u}}{\partial t}\right]_{k+1} = \mathbf{v}^{k+1} \text{ on } \Gamma_{fs}$$

where

$$\mathbb{V}_h \subset H^1(\widehat{\Omega})^3, \mathbb{Q}_h \subset L^2(\widehat{\Omega}), \mathbb{V}_h^0 = \{\mathbf{v} \in \mathbb{V}_h \,:\, \mathbf{v}|_{\Gamma_{s0} \cup \Gamma_{f0}} = \mathbf{0}\}, \mathbb{V}_h^{00} = \{\mathbf{v} \in \mathbb{V}_h^0 \,:\, \mathbf{v}|_{\Gamma_{fs}} = \mathbf{0}\}$$

$$\left[\frac{\partial \mathbf{f}}{\partial t}\right]_{k+1} := \frac{3\mathbf{f}^{k+1} - 4\mathbf{f}^k + \mathbf{f}^{k-1}}{2\Delta t}$$

Numerical scheme

$$\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla\psi \, \mathrm{d}\Omega +$$
$$\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k) \left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]}_k \right) \psi \, \mathrm{d}\Omega +$$
$$\int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\widetilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\widetilde{\mathbf{u}}^k} \psi \, \mathrm{d}\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) : \nabla\psi \, \mathrm{d}\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0$$

$$J_k := J(\widetilde{\mathbf{u}}^k), \quad \widetilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}}\mathbf{v} := \{\nabla \mathbf{v} \mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

$$\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \,\mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla \psi \,\mathrm{d}\Omega +$$
$$\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \,\mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k) \left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]}_k\right) \psi \,\mathrm{d}\Omega +$$
$$\int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\widetilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\widetilde{\mathbf{u}}^k} \psi \,\mathrm{d}\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) : \nabla \psi \,\mathrm{d}\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0$$

$$\int_{\Omega_s} \left[\frac{\partial \mathbf{u}}{\partial t}\right]_{k+1} \phi \,\mathrm{d}\Omega - \int_{\Omega_s} \mathbf{v}^{k+1} \phi \,\mathrm{d}\Omega + \int_{\Omega_f} G(\mathbf{u}^{k+1}) \phi \,\mathrm{d}\Omega = 0 \quad \forall \phi \in \mathbb{V}_h^{00}$$

$$J_k := J(\widetilde{\mathbf{u}}^k), \quad \widetilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}}\mathbf{v} := \{\nabla \mathbf{v} \mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

$$\int_{\Omega_s} \rho_s \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla\psi \, \mathrm{d}\Omega +$$

$$\int_{\Omega_f} \rho_f J_k \left[\frac{\partial \mathbf{v}}{\partial t}\right]_{k+1} \psi \, \mathrm{d}\Omega + \int_{\Omega_f} \rho_f J_k \nabla \mathbf{v}^{k+1} \mathbf{F}^{-1}(\widetilde{\mathbf{u}}^k) \left(\widetilde{\mathbf{v}}^k - \widetilde{\left[\frac{\partial \mathbf{u}}{\partial t}\right]}_k\right) \psi \, \mathrm{d}\Omega +$$

$$\int_{\Omega_f} 2\mu_f J_k \mathbf{D}_{\widetilde{\mathbf{u}}^k} \mathbf{v}^{k+1} : \mathbf{D}_{\widetilde{\mathbf{u}}^k} \psi \, \mathrm{d}\Omega - \int_{\Omega} p^{k+1} J_k \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) : \nabla\psi \, \mathrm{d}\Omega = 0 \quad \forall \psi \in \mathbb{V}_h^0$$

$$\int_{\Omega_s} \left[\frac{\partial \mathbf{u}}{\partial t}\right]_{k+1} \phi \, \mathrm{d}\Omega - \int_{\Omega_s} \mathbf{v}^{k+1} \phi \, \mathrm{d}\Omega + \int_{\Omega_f} G(\mathbf{u}^{k+1}) \phi \, \mathrm{d}\Omega = 0 \quad \forall \phi \in \mathbb{V}_h^{00}$$

$$\int_{\Omega_f} J_k \nabla \mathbf{v}^{k+1} : \mathbf{F}^{-T}(\widetilde{\mathbf{u}}^k) q \, \mathrm{d}\Omega = 0 \quad \forall \ q \in \mathbb{Q}_h$$

$$J_k := J(\widetilde{\mathbf{u}}^k), \quad \widetilde{\mathbf{f}}^k := 2\mathbf{f}^k - \mathbf{f}^{k-1}, \quad \mathbf{D}_{\mathbf{u}}\mathbf{v} := \{\nabla \mathbf{v} \mathbf{F}^{-1}(\mathbf{u})\}_s, \quad \{\mathbf{A}\}_s := \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

$$\dots + \int_{\Omega_s} J_k \mathbf{F}(\widetilde{\mathbf{u}}^k)\mathbf{S}(\mathbf{u}^{k+1}, \widetilde{\mathbf{u}}^k) : \nabla\psi \,\mathrm{d}\Omega + \dots$$

- St. Venant–Kirchhoff model (geometrically nonlinear):

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \lambda_s \mathtt{tr}(\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2))\mathbf{I} + 2\mu_s \mathbf{E}(\mathbf{u}_1, \mathbf{u}_2);$$

$$\mathbf{E}(\mathbf{u}_1, \mathbf{u}_2) = \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2) - \mathbf{I}\}_s$$

- inc. Blatz–Ko model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s(\mathtt{tr}(\{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s)\mathbf{I} - \{\mathbf{F}(\mathbf{u}_1)^T \mathbf{F}(\mathbf{u}_2)\}_s)$$

- inc. Neo-Hookean model:

$$\mathbf{S}(\mathbf{u}_1, \mathbf{u}_2) = \mu_s \mathbf{I};\ \mathbf{F}(\widetilde{\mathbf{u}}^k) \to \mathbf{F}(\mathbf{u}^{k+1})$$

$$\{\mathbf{A}\}_s := \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

Numerical scheme

The scheme

- provides strong coupling on interface
- semi-implicit
- produces one linear system per time step
- second order in time

Numerical scheme

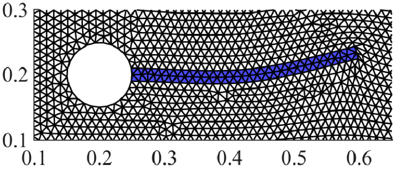
The scheme

- provides strong coupling on interface
- semi-implicit
- produces one linear system per time step
- second order in time
- unconditionally stable (no CFL restriction), proved with assumptions:
  - 1st order in time
  - St. Venant–Kirchhoff inc./comp. (experiment: Neo-Hookean inc./comp.)
  - extension of $\mathbf{u}$ to Ω_f guarantees $J_k > 0$
  - Δt is not large

A.Lozovskiy, M.Olshanskii, V.Salamatova, Yu.Vassilevski. An unconditionally stable semi-implicit FSI finite element method. *Comput.Methods Appl.Mech.Engrg., 297, 2015*

2D validation: FSI3 2D benchmark problem

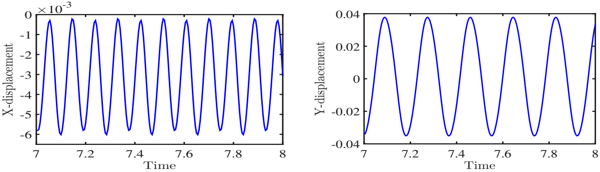
S. Turek and J. Hron. Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. In: *Fluid-structure interaction*, Springer Berlin Heidelberg, 371–385, 2006.



- fluid: 2D Navier-Stokes
- stick: Saint Venant-Kirchoff constitutive relation
- inflow: parabolic profile
- outflow: “do-nothing”
- rigid walls: Dirichlet

2D validation: FSI3 2D benchmark problem

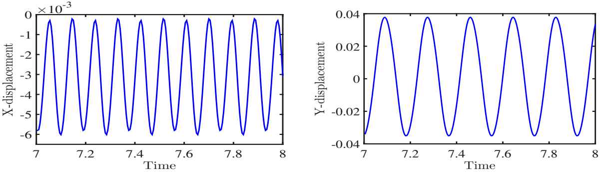
S. Turek and J. Hron. Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. In: *Fluid-structure interaction*, Springer Berlin Heidelberg, 371–385, 2006.



- Fortran open source software `Ani2D`
  Advanced numerical instruments 2D, `http://sf.net/p/ani2d/`

2D validation: FSI3 2D benchmark problem

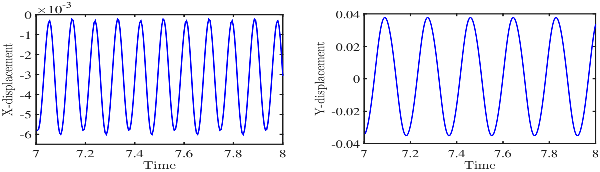
S. Turek and J. Hron. Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. In: *Fluid-structure interaction*, Springer Berlin Heidelberg, 371–385, 2006.



- Fortran open source software `Ani2D`
  Advanced numerical instruments 2D, `http://sf.net/p/ani2d/`
- Displacement in fluid equation: Laplace → mesh tangling, heterogeneous elastisity (more stiff close-to-stick) → OK

2D validation: FSI3 2D benchmark problem

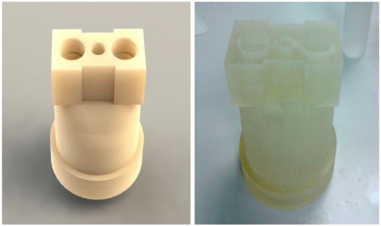
S. Turek and J. Hron. Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. In: *Fluid-structure interaction*, Springer Berlin Heidelberg, 371–385, 2006.

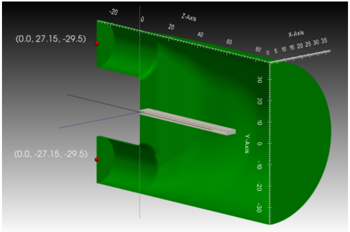


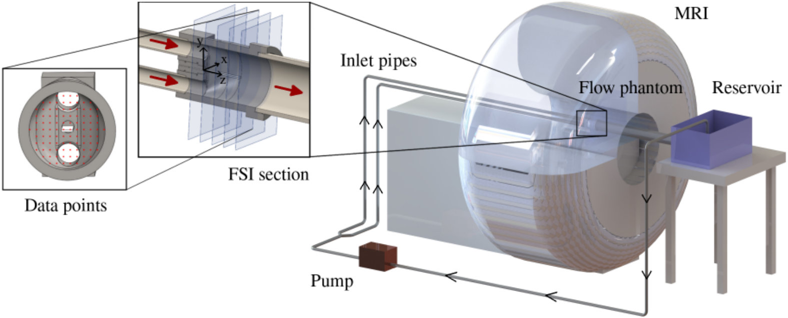
- Fortran open source software `Ani2D`
  Advanced numerical instruments 2D, `http://sf.net/p/ani2d/`
- Displacement in fluid equation: Laplace $\rightarrow$ mesh tangling, heterogeneous elastisity (more stiff close-to-stick) $\rightarrow$ OK
- Simulations are fast, stable, reproduce
  - displacements
  - drag, lift
  - Strouhal numbers

3D benchmark: unsteady flow around silicon filament

A. Hessenthaler et al. Experiment for validation of fluid-structure interaction models and algorithms. *Int.J.Numer.Meth.Biomed.Engng.*,2016.







Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain Ω_f

Let $\boldsymbol{\xi}$ mapping Ω_f to $\Omega_f(t)$, $\mathbf{F} = \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J = \det(\mathbf{F})$ be given

Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain Ω_f

Let $\boldsymbol{\xi}$ mapping Ω_f to $\Omega_f(t)$, $\mathbf{F} = \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J = \det(\mathbf{F})$ be given

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = (J\rho_f)^{-1}\mathrm{div}\left(J\boldsymbol{\sigma}_f\mathbf{F}^{-T}\right) - \nabla\mathbf{v}\left(\mathbf{F}^{-1}\left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\right)\right) \quad \text{in } \Omega_f$$

Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain Ω_f

Let $\boldsymbol{\xi}$ mapping Ω_f to $\Omega_f(t)$, $\mathbf{F} = \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J = \det(\mathbf{F})$ be given

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = (J\rho_f)^{-1}\mathrm{div}\left(J\boldsymbol{\sigma}_f\mathbf{F}^{-T}\right) - \nabla\mathbf{v}\left(\mathbf{F}^{-1}\left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\right)\right) \quad \text{in } \Omega_f$$

Fluid incompressibility

$$\mathrm{div}\left(J\mathbf{F}^{-1}\mathbf{v}\right) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J\nabla\mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain Ω_f

Let $\boldsymbol{\xi}$ mapping Ω_f to $\Omega_f(t)$, $\mathbf{F} = \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J = \det(\mathbf{F})$ be given

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = (J\rho_f)^{-1}\mathrm{div}\,(J\boldsymbol{\sigma}_f\mathbf{F}^{-T}) - \nabla\mathbf{v}\left(\mathbf{F}^{-1}\left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\right)\right) \quad \text{in } \Omega_f$$

Fluid incompressibility

$$\mathrm{div}\,(J\mathbf{F}^{-1}\mathbf{v}) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J\nabla\mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f\mathbf{I} + \mu_f((\nabla\mathbf{v})\mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla\mathbf{v})^T) \quad \text{in } \Omega_f$$

Incompressible fluid flow in a moving domain

Navier-Stokes equations in reference domain Ω_f

Let $\boldsymbol{\xi}$ mapping Ω_f to $\Omega_f(t)$, $\mathbf{F} = \nabla\boldsymbol{\xi} = \mathbf{I} + \nabla\mathbf{u}$, $J = \det(\mathbf{F})$ be given

Dynamic equations

$$\frac{\partial \mathbf{v}}{\partial t} = (J\rho_f)^{-1}\mathrm{div}\,(J\boldsymbol{\sigma}_f\mathbf{F}^{-T}) - \nabla\mathbf{v}\left(\mathbf{F}^{-1}\left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\right)\right) \quad \text{in } \Omega_f$$

Fluid incompressibility

$$\mathrm{div}\,(J\mathbf{F}^{-1}\mathbf{v}) = 0 \quad \text{in } \Omega_f \quad \text{or} \quad J\nabla\mathbf{v} : \mathbf{F}^{-T} = 0 \quad \text{in } \Omega_f$$

Constitutive relation for the fluid stress tensor

$$\boldsymbol{\sigma}_f = -p_f\mathbf{I} + \mu_f((\nabla\mathbf{v})\mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla\mathbf{v})^T) \quad \text{in } \Omega_f$$

Mapping $\boldsymbol{\xi}$ does not define material trajectories $\rightarrow$ quasi-Lagrangian formulation

Finite element scheme

Find $\{\mathbf{v}_h^k, p_h^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$ satisfying b.c.

("do nothing" $\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{n} = 0$ or no-penetration no-slip $\mathbf{v}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$)

Finite element scheme

Find $\{\mathbf{v}_h^k, p_h^k\} \in \mathbb{V}_h \times \mathbb{Q}_h$ satisfying b.c.

("do nothing" $\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{n} = 0$ or no-penetration no-slip $\mathbf{v}^k = (\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1})/\Delta t$)

$$\int_{\Omega_f} J_k \frac{\mathbf{v}_h^k - \mathbf{v}_h^{k-1}}{\Delta t} \cdot \psi \, \mathrm{d}\mathbf{x} + \int_{\Omega_f} J_k \nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \left(\mathbf{v}_h^{k-1} - \frac{\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1}}{\Delta t} \right) \cdot \psi \, \mathrm{d}\mathbf{x} -$$

$$\int_{\Omega_f} J_k p_h^k \mathbf{F}_k^{-T} : \nabla \psi \, \mathrm{d}\mathbf{x} + \int_{\Omega_f} J_k q \mathbf{F}_k^{-T} : \nabla \mathbf{v}_h^k \, \mathrm{d}\mathbf{x} +$$

$$\int_{\Omega_f} \nu J_k (\nabla \mathbf{v}_h^k \mathbf{F}_k^{-1} \mathbf{F}_k^{-T} + \mathbf{F}_k^{-T} (\nabla \mathbf{v}_h^k)^T \mathbf{F}_k^{-T}) : \nabla \psi \, \mathrm{d}\mathbf{x} = 0$$

for all ψ and q from the appropriate FE spaces

Finite element scheme

The scheme

- semi-implicit
- produces one linear system per time step
- first order in time (may be generalized to the second order)

Finite element scheme

The scheme

- semi-implicit
- produces one linear system per time step
- first order in time (may be generalized to the second order)
- unconditionally stable (no CFL restriction), proved with assumptions:
  - $\inf_Q J \geq c_J > 0, \quad \sup_Q(\|\mathbf{F}\|_F + \|\mathbf{F}^{-1}\|_F) \leq C_F$
  - LBB-stable pairs (e.g. P_2/P_1 or P_2/P_0)
  - Δt is not large

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

Energy equality for the weak solution

Let $\partial\Omega(t) = \partial\Omega^{ns}(t)$ and $\boldsymbol{\xi}_t$ be given on $\partial\Omega^{ns}(t)$. Then there exists $\mathbf{v}_1 \in C^1(Q)^d$, $\mathbf{v}_1 = \boldsymbol{\xi}_t$, $\operatorname{div}(J\mathbf{F}^{-1}\mathbf{v}_1) = 0$ [Miyakawa1982]

and we can decompose the solution $\mathbf{v} = \mathbf{w} + \mathbf{v}_1$, $\mathbf{w} = 0$ on $\partial\Omega^{ns}$

Energy equality for the weak solution

Let $\partial\Omega(t) = \partial\Omega^{ns}(t)$ and $\boldsymbol{\xi}_t$ be given on $\partial\Omega^{ns}(t)$. Then there exists $\mathbf{v}_1 \in C^1(Q)^d$, $\mathbf{v}_1 = \boldsymbol{\xi}_t$, $\operatorname{div}(J\mathbf{F}^{-1}\mathbf{v}_1) = 0$ [Miyakawa1982]

and we can decompose the solution $\mathbf{v} = \mathbf{w} + \mathbf{v}_1$, $\mathbf{w} = 0$ on $\partial\Omega^{ns}$

Energy balance for w:

$$\underbrace{\frac{1}{2}\frac{d}{dt}\|J^{\frac{1}{2}}\mathbf{w}\|^2}_{\text{variation of kinetic energy}} \quad \underbrace{+2\nu\|J^{\frac{1}{2}}\mathbf{D}_\xi(\mathbf{w})\|^2}_{\text{energy of viscous dissipation}} \quad \underbrace{+(J(\nabla\mathbf{v}_1\mathbf{F}^{-1}\mathbf{w}),\mathbf{w})}_{\text{intensification due to b.c.}} \quad \underbrace{=(\widetilde{\mathbf{f}},\mathbf{w})}_{\text{work of ext. forces}}$$

$$\mathbf{D}_\xi(\mathbf{v}) = \tfrac{1}{2}(\nabla\mathbf{v}\mathbf{F}^{-1} + \mathbf{F}^{-T}(\nabla\mathbf{v})^T)$$

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

Stability estimate for the FE solution

Energy equality for $\mathbf{w}_h = \mathbf{v}_h - \mathbf{v}_{1,h}$:

$$\underbrace{\frac{1}{2\Delta t}\left(\|J_k^{\frac{1}{2}}\mathbf{w}_h^k\|^2 - \|J_{k-1}^{\frac{1}{2}}\mathbf{w}_h^{k-1}\|^2\right)}_{\text{variation of kinetic energy}} \underbrace{+2\nu\left\|J_k^{\frac{1}{2}}\mathbf{D}_k(\mathbf{w}_h^k)\right\|^2}_{\text{energy of viscous dissipation}} \underbrace{+\frac{(\Delta t)}{2}\left\|J_{k-1}^{\frac{1}{2}}[\mathbf{w}_h]_t^k\right\|^2}_{O(\Delta t)\text{ dissipative term}}$$

$$\underbrace{+(J_k(\nabla\mathbf{v}_1^k\mathbf{F}_k^{-1})\mathbf{w}_h^k,\mathbf{w}_h^k)}_{\text{intensification due to b.c.}} = \underbrace{(\widetilde{\mathbf{f}}^k,\mathbf{w}_h^k)}_{\text{work of ext. forces}}$$

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

Stability estimate for the FE solution

Stability estimate:

$C_1\|\nabla \mathbf{v}_1^k\| \le \nu/2$:

$$\frac{1}{2}\|\mathbf{w}_h^n\|_n^2 + \nu\sum_{k=1}^{n}\Delta t\|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \le \frac{1}{2}\|\mathbf{w}_0\|_0^2 + C\sum_{k=1}^{n}\Delta t\|\widetilde{\mathbf{f}}^k\|^2$$

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

Stability estimate for the FE solution

Stability estimate:

$C_1\|\nabla \mathbf{v}_1^k\| \le \nu/2$:

$$\frac{1}{2}\|\mathbf{w}_h^n\|_n^2 + \nu\sum_{k=1}^{n}\Delta t\|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \le \frac{1}{2}\|\mathbf{w}_0\|_0^2 + C\sum_{k=1}^{n}\Delta t\|\widetilde{\mathbf{f}}^k\|^2$$

$C_1\|\nabla \mathbf{v}_1^k\| > \nu/2$:

$$\frac{1}{2}\|\mathbf{w}_h^n\|_n^2+\nu\sum_{k=1}^{n}\Delta t\|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \le e^{\frac{2C_2}{\alpha}T}\left(\frac{1}{2}\|\mathbf{w}_0\|_0^2 + C\sum_{k=1}^{n}\Delta t\|\widetilde{\mathbf{f}}^k\|^2\right),$$

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

Stability estimate for the FE solution

Stability estimate:

$C_1\|\nabla \mathbf{v}_1^k\| \le \nu/2$:

$$\frac{1}{2}\|\mathbf{w}_h^n\|_n^2 + \nu\sum_{k=1}^{n}\Delta t\|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \le \frac{1}{2}\|\mathbf{w}_0\|_0^2 + C\sum_{k=1}^{n}\Delta t\|\widetilde{\mathbf{f}}^k\|^2$$

$C_1\|\nabla \mathbf{v}_1^k\| > \nu/2$:

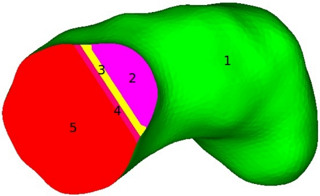
$$\frac{1}{2}\|\mathbf{w}_h^n\|_n^2 + \nu\sum_{k=1}^{n}\Delta t\|\mathbf{D}_k(\mathbf{w}_h^k)\|_k^2 \le e^{\frac{2C_2}{\alpha}T}\left(\frac{1}{2}\|\mathbf{w}_0\|_0^2 + C\sum_{k=1}^{n}\Delta t\|\widetilde{\mathbf{f}}^k\|^2\right),$$

$$\text{if } (1 - 2C_2\Delta t) = \alpha > 0$$

A.Danilov, A.Lozovskiy, M.Olshanskii, Yu.Vassilevski. A finite element method for the Navier-Stokes equations in moving domain with application to hemodynamics of the left ventricle. *Russian J. Numer. Anal. Math. Modelling, 32, 2017*

Application to the left ventricle hemodynamics

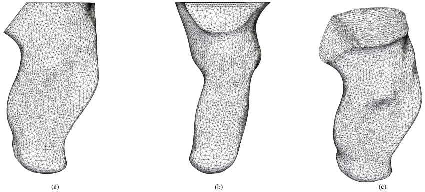
- Boundary conditions: no-penetration no-slip $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ and "do-nothing"



Application to the left ventricle hemodynamics

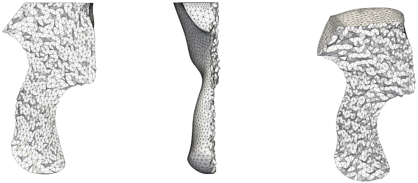
- Boundary conditions: no-penetration no-slip $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ and "do-nothing"
- Computational meshes contain 14033 nodes, 69257 tetrahedra, 88150 edges (320k unknowns)

100 meshes with varying node positions



Application to the left ventricle hemodynamics

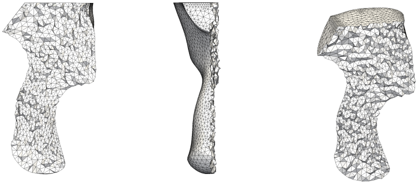
- Boundary conditions: no-penetration no-slip $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ and "do-nothing"
- Computational meshes contain 14033 nodes, 69257 tetrahedra, 88150 edges (320k unknowns) 100 meshes with varying node positions
- Time step between given frames 0.0127 s $\implies$ interpolate intermediate meshes with $\Delta t = 0.0127/20$



(a) (b) (c)

Application to the left ventricle hemodynamics

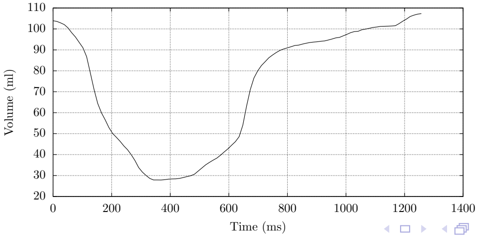
- Boundary conditions: no-penetration no-slip $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ and "do-nothing"
- Computational meshes contain 14033 nodes, 69257 tetrahedra, 88150 edges (320k unknowns) 100 meshes with varying node positions
- Time step between given frames 0.0127 s $\Longrightarrow$ interpolate intermediate meshes with $\Delta t = 0.0127/20$
- $\rho_f = 10^3$ kg/m^3, $\mu_f = 4 \cdot 10^{-2}$ Pa $\cdot$s $(= 10\mu_{blood})$



(a) (b) (c)

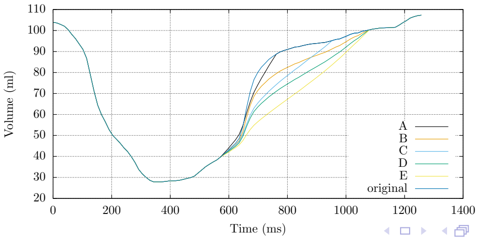
Application to the left ventricle hemodynamics

- Boundary conditions: no-penetration no-slip $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ and "do-nothing"
- Computational meshes contain 14033 nodes, 69257 tetrahedra, 88150 edges (320k unknowns) 100 meshes with varying node positions
- Time step between given frames 0.0127 s $\implies$ interpolate intermediate meshes with $\Delta t = 0.0127/20$
- $\rho_f = 10^3$ kg/m^3, $\mu_f = 4 \cdot 10^{-2}$ Pa $\cdot$s $(= 10\mu_{blood})$
- Large volume rate $\implies$ extremely high influx velocities at the diastole phase

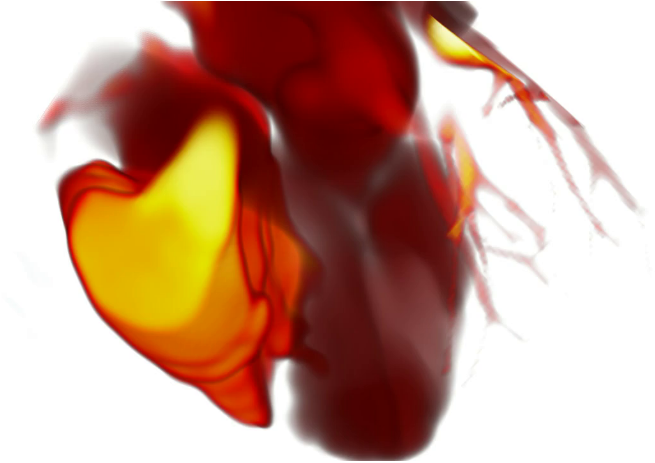


Application to the left ventricle hemodynamics

- Boundary conditions: no-penetration no-slip $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ and "do-nothing"
- Computational meshes contain 14033 nodes, 69257 tetrahedra, 88150 edges (320k unknowns)   100 meshes with varying node positions
- Time step between given frames 0.0127 s $\implies$ interpolate intermediate meshes with $\Delta t = 0.0127/20$
- $\rho_f = 10^3$ kg/m^3, $\mu_f = 4 \cdot 10^{-2}$ Pa ·s $(= 10\mu_{blood})$
- Large volume rate $\implies$ extremely high influx velocities at the diastole phase
- $\implies$ need time smoothing for the wall displacements



Application to the left ventricle hemodynamics



Conclusions

- We proposed unconditionally stable semi-implicit FE schemes for FSI problem and NS eqs in moving domain

Conclusions

- We proposed unconditionally stable semi-implicit FE schemes for FSI problem and NS eqs in moving domain
- The FSI scheme can incorporate diverse elasticity models

Conclusions

- We proposed unconditionally stable semi-implicit FE schemes for FSI problem and NS eqs in moving domain
- The FSI scheme can incorporate diverse elasticity models
- Only one linear system is solved per time step

Conclusions

- We proposed unconditionally stable semi-implicit FE schemes for FSI problem and NS eqs in moving domain
- The FSI scheme can incorporate diverse elasticity models
- Only one linear system is solved per time step
- Ventricle hemodynamics: the NS scheme suffers from convection instability due to large inflows

Conclusions

- We proposed unconditionally stable semi-implicit FE schemes for FSI problem and NS eqs in moving domain
- The FSI scheme can incorporate diverse elasticity models
- Only one linear system is solved per time step
- Ventricle hemodynamics: the NS scheme suffers from convection instability due to large inflows
- We work on its stabilization