Weyl metrics and wormholes

Mikhail S. Volkov

LMPT, University of Tours, FRANCE

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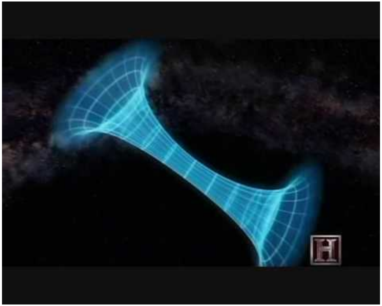
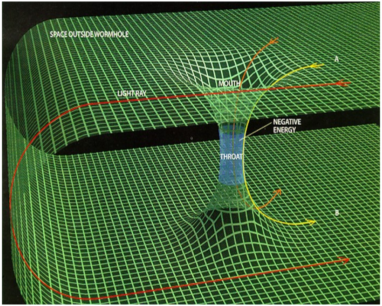
- Introduction
- I. Gravitating scalar field
- II. Vacuum wormholes
- III. Zero mass limit of Kerr spacetime is a wormhole

Wormholes – spacetime bridges

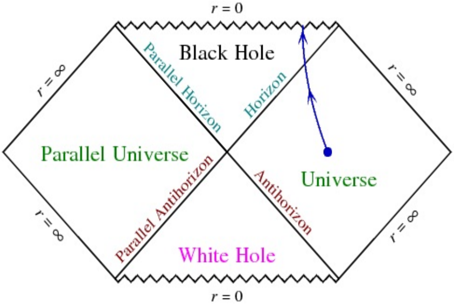




Wormholes interpolate between different universes or between different parts of the same universe. Could supposedly be used for interstellar and time travels.

Some history

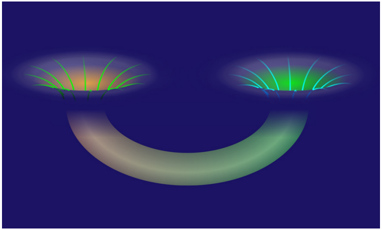
- /Einstein-Rosen, 1935/ – Schwarzschild black hole has two exterior regions connected by a bridge. The ER bridge is spacelike and cannot be traversed by classical objects.



- /Maldacena-Susskind, 2013/ – the ER bridge may connect quantum particles to produce quantum entanglement and the Einstein-Pololsky-Rosen (EPR) effect, hence ER=EPR.

Some history

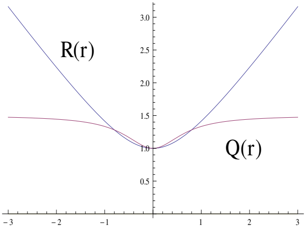
- /Wheeler, 1957/ wormholes may provide geometric models of elementary particles – handles of space trapping inside an electric flux.



- /Misner, 1960/ Wormholes can describe initial data for the Einstein equations. The time evolution of these data corresponds to the black hole collisions of the type observed in the recent GW150914 event.
- /Morris, Thorn, Yurtsever, 1988/ wormholes traversable by classical object may be supported by vacuum polarisation.

Can wormholes be solutions of Einstein equations ?

$$ds^2 = -Q^2(r)dt^2 + dr^2 + R^2(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$



$G_{\mu\nu} = T_{\mu\nu} \Rightarrow$ energy $\rho = -T_0^0$ and pressure $p = T_r^r$ fulfill

$$\rho + p = -2\frac{R''}{R} < 0, \qquad p = -\frac{1}{R^2} < 0.$$

$\Rightarrow$ the Null Energy Condition (NEC) must be violated.
$/\, T_{\mu\nu}v^\mu v^\nu = R_{\mu\nu}v^\mu v^\nu \geq 0$ for any null $v^\mu /$

NEC violation

The general case without symmetry $\Rightarrow$ topological censorship: compact two-surface of minimal area can exit if only NEC is violated /Friedman, Schleich, Witt, 1993/ $\Rightarrow$ traversable wormholes are possible if only energy is negative, for example, due to

- vacuum polarization
- exotic matter: phantom fields, etc.

Wormholes may exist in alternative gravity models:

- Gauss-Bonnet brainworld, etc.
- theories with non-minimally coupled fields (Horndeski)
- massive (bi)gravity

Best known example – phantom-supported wormhole

$$\mathcal{L} = R + 2(\partial\psi)^2$$

Bronnikov-Ellis wormhole:

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2)(d\vartheta^2 + \sin^2\vartheta d\varphi)^2, \quad \psi = \arctan\left(\frac{r}{a}\right);$$

$$r \in (-\infty, \infty)$$

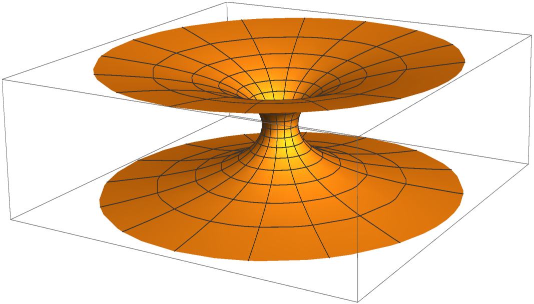


Figure: *Isometric embedding of the equatorial section of the BE wormhole to the 3-dimensional Euclidean space*

Wormholes without phantom field

Write a phantom field solution in the Weyl form,

$$ds^2 = -e^{2U}dt^2 + e^{2U}\left(e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right), \qquad \psi = \psi$$

A new solution of the same form is obtained by swapping

$$U \leftrightarrow \psi, \qquad k \rightarrow -k$$

hence by setting

$$U_{\mathrm{new}} = \psi, \qquad \psi_{\mathrm{new}} = U, \qquad k_{\mathrm{new}} = -k$$

The BE wormhole is ultrastatic, $U = 0$, hence the new solution is vacuum, $\psi_{\mathrm{new}} = 0$, but it keeps the original topology with two asymptotic regions – wormhole. The negative energy is hidden in the singularity.

I. Gravitating scalar field

Axial symmetry

$$\mathcal{L} = R - 2\epsilon\,(\partial\Phi)^2$$

- $\epsilon = +1$: ordinary scalar $\Phi \equiv \phi$
- $\epsilon = -1$: phantom $\Phi \equiv \psi$

Static, axially symmetric system

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left\{e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right\}$$

where U, k, Φ depend on ρ, z.

Field equations

$$
\begin{aligned}
\frac{\partial^2 U}{\partial \rho^2} &+ \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = 0, \\
\frac{\partial^2 \Phi}{\partial \rho^2} &+ \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \\
\frac{\partial k}{\partial \rho} &= \rho\left[\left(\frac{\partial U}{\partial \rho}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2 + \epsilon\left(\frac{\partial \Phi}{\partial \rho}\right)^2 - \epsilon\left(\frac{\partial \Phi}{\partial z}\right)^2\right], \\
\frac{\partial k}{\partial z} &= 2\rho\left[\frac{\partial U}{\partial \rho}\frac{\partial U}{\partial z} + \epsilon\,\frac{\partial \Phi}{\partial \rho}\frac{\partial \Phi}{\partial z}\right].
\end{aligned}
$$

Target space symmetries

preserve spherical symmetry:

rotations $$\begin{pmatrix} U \\ \phi \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} U \\ \phi \end{pmatrix}, \qquad k \to k$$

boosts $$\begin{pmatrix} U \\ \psi \end{pmatrix} \to \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} U \\ \psi \end{pmatrix}, \qquad k \to k$$

interchange BE wormhole and ring wormhole:

swap $$U \leftrightarrow \psi, \qquad k \to -k$$

do not intermix scalar field and gravity amplitudes:

scaling $$U \to \lambda U, \qquad k \to \lambda^2 k, \qquad \Phi \to \lambda \Phi$$

tachyon: $$U \to \ln\rho - U, \qquad k \to k - 2U + \ln\rho, \qquad \Phi \to \Phi$$

Acting with this on vacuum metrics yields new solutions.

Simplest vacuum Weyl metrics

One rod – Schwarzschild

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left\{e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right\}$$

with

$$\begin{aligned} U(\rho, z) &= \frac{1}{2}\ln\left(\frac{R-m}{R+m}\right) = -\frac{1}{2}\int_{-m}^{m}\frac{d\zeta}{\sqrt{\rho^2 + (z-\zeta)^2}} \\ k(\rho, z) &= \frac{1}{2}\ln\left(\frac{R^2 - m^2}{R_+ R_-}\right) \end{aligned}$$

where

$$R = \frac{1}{2}(R_+ + R_-), \qquad R_\pm = \sqrt{\rho^2 + (z \pm m)^2}.$$

U is the Newtonian potential of a massive rod of length $2m$ with mass density $1/2$.

Two rods

$$U = U_1 + U_2, \qquad k = k_1 + k_2 + k_{12}\,,$$

where (with $a = 1, 2$)

$$\begin{aligned} U_a &= \frac{1}{2}\ln\left(\frac{R_a - m_a}{R_a + m_a}\right), \qquad k_a = \frac{1}{2}\ln\left(\frac{(R_a)^2 - (m_a)^2}{R_{a+}R_{a-}}\right), \\ k_{12} &= \frac{1}{2}\ln\left(\frac{(R_{1+}R_{2-} + z_{1+}z_{2-} + \rho^2)(R_{1-}R_{2+} + z_{1-}z_{2+} + \rho^2)}{(R_{1+}R_{2+} + z_{1+}z_{2+} + \rho^2)(R_{1-}R_{2-} + z_{1-}z_{2-} + \rho^2)}\right), \end{aligned}$$

with

$$z_{a\pm} = z - z_a \pm m_a, \quad R_{a\pm} = \sqrt{\rho^2 + (z_{a\pm})^2}\,, \quad R_a = \frac{1}{2}(R_{a+} + R_{a-})$$

$k \neq 0$ on the part of symmetry axis between the rods – strut /Israel and Khan 1964/ Similarly for many rods.

Point masses

$$U = -\frac{m}{R}\,, \qquad k = -\frac{m^2\rho^2}{2R^4}\,,$$

with $R = \sqrt{\rho^2 + z^2}$. For two masses $m_\pm$ at $z = \pm m$ one has

$$\begin{aligned} U &= -\frac{m_+}{R_+} - \frac{m_-}{R_-}, \\ k &= -\frac{m_+^2\rho^2}{2(R_+)^4} - \frac{m_-^2\rho^2}{2(R_-)^4} + \frac{m_+ m_-}{2m^2}\left(\frac{\rho^2 + z^2 - m^2}{R_+ R_-} - 1\right) \end{aligned}$$

with $R_\pm = \sqrt{\rho^2 + (z \pm m)^2}$. /Chazy, Curzon 1924/

Summary of part I

- Applying the target space dualities to the vacuum Weyl metric produced by massive rods and points gives all known and also large classes of new static solutions for gravity-coupled scalar field.
- These are, for example, the Fisher-Janis-Robinson-Winicour solutions and their generalizations to axially symmetric case.
- These are, for example, the Bronnikov-Ellis wormholes and their generalizations to axially symmetric case.
- Many more other new solutions.

II. Vacuum wormholes

One rod – Schwarzschild

- Scaling $U \to \lambda U$, $k \to \lambda^2 k$ *(prolate vacuum metrics)*

$$
\begin{aligned}
U(\rho, z) &= \frac{\lambda}{2} \ln\left(\frac{R-m}{R+m}\right), \qquad k(\rho, z) = \frac{\lambda^2}{2} \ln\left(\frac{R^2 - m^2}{R_+ R_-}\right) \\
R &= \frac{1}{2}(R_+ + R_-), \qquad R_\pm = \sqrt{\rho^2 + (z \pm m)^2}
\end{aligned}
$$

- $m \to ia$, $\lambda \to i\sigma$ *(oblate vacuum metrics)*

$$
\begin{aligned}
U &= \sigma \arctan\left(\frac{X}{a}\right), \qquad k = \frac{\sigma^2}{2} \ln\left(\frac{X^2 + Y^2}{X^2 + a^2}\right) \\
X + iY &= \sqrt{\rho^2 + (z + ia)^2}
\end{aligned}
$$

two sheets of the square root $\Rightarrow$ double-sheeted topology with two asymptotically flat regions $\Rightarrow$ wormhole. For $\sigma = 1$ the swap $U \leftrightarrow \psi$, $k \to -k$ gives the Bronnikov-Ellis wormhole.

Global wormhole coordinates

$$z = r\cos\vartheta, \qquad \rho = \sqrt{r^2 + a^2}\,\sin\vartheta$$

with $r \in (-\infty, \infty)$ (double covering) yields

$$\begin{aligned} ds^2 &= -e^{2U}dt^2 + e^{-2U}dl^2, \qquad\qquad U = \sigma \arctan\left(\frac{r}{a}\right), \\ dl^2 &= \left(\frac{r^2 + a^2\cos^2\vartheta}{r^2 + a^2}\right)^{1+\sigma^2} \left[dr^2 + (r^2 + a^2)d\vartheta^2\right] \\ &+ (r^2 + a^2)\sin^2\vartheta d\varphi^2 \,. \end{aligned}$$

Close to the axis $\cos\vartheta \approx 1$, taking $\sigma \to 0$ gives wormhole metric

$$ds^2 = -dt^2 + dx^2 + (x^2 + a^2)d\Omega^2$$

Wormhole throat is at $r = 0$. The Weyl coordinates (ρ, z) cover either the $r < 0$ part or the $r > 0$ wormhole parts.

Wormhole topology

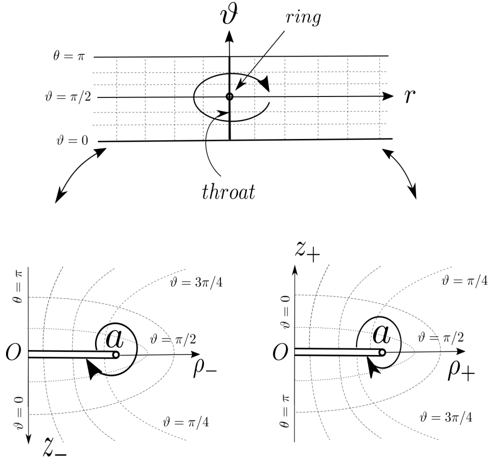


Figure: The r, ϑ coordinates cover the whole of the manifold, each Weyl chart covers only a half. The Weyl charts have branch cuts. A winding around the ring in the x, ϑ coordinates corresponds to two windings in Weyl coordinates.

Ring wormhole

Metric is singular at the ring in the equatorial plane at $r = 0$ $\vartheta = \pi/2$. In its vicinity

$$ds^2 = -dt^2 + dR^2 + R^2\alpha^2 + a^2 d\varphi^2 + \ldots$$

with $\alpha \in [0, (2 + \sigma^2)2\pi)$ $\Rightarrow$ a negative angle deficit

$$\delta = -(\sigma^2 + 1)2\pi$$

$\Rightarrow$ a conical singularity generated by an infinitely thin ring of radius a and of *negative* tension (energy per unit length)

$$T = -\frac{(1 + \sigma^2)c^4}{4G}$$

$\Rightarrow$ the wormhole is supported by a negative tension ring.

Ring wormhole with locally flat geometry

In the limit $\sigma \to 0$ one has

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2\vartheta}{r^2 + a^2}\right)\left[dr^2 + (r^2 + a^2)d\vartheta^2\right] \\ +(r^2 + a^2)\sin^2\vartheta d\varphi^2$$

while in Weyl coordinates the metric is manifestly flat,

$$ds^2 \;\; = \;\; -dt^2 + d\rho^2 + dz^2 + \rho^2 d\varphi^2.$$

However, the topology is still non-trivial since for $r \in (-\infty, \infty)$ one needs two (ρ, z) patches, one for $r > 0$ and the other for $r < 0$, to cover the manifold. Therefore, the winding angle around the ring core ranges from zero to 4π hence the ring is still there and has the tension

$$T = -c^4/4G$$

the curvature is zero everywhere outside the ring.

Geodesics

Geodesics are straight lines. Those which miss the ring always stay at the same chart. Those threading the ring pass to the other chart and become invisible – the ring literally creates a hole in flat space.

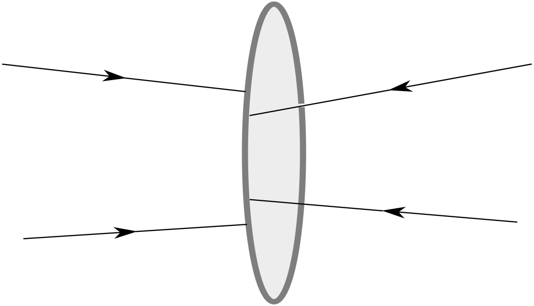


Figure: Particles entering the ring are not seen coming out from the other side

Energy

To create a ring of radius R one needs the negative energy

$$E = 2\pi RT = -2\pi R\frac{c^4}{4G}$$

To create a ring of radius $R = 1$ metre one needs a negative energy equivalent to the mass of Jupiter.

Small rings can probably appear and disappear in quantum fluctuations. Particles passing through the ring during its existence will disappear – loss of quantum coherence.

The ring can probably be replaced by a thin tours. The negative energy could probably be associated to quantum fluctuations inside the torus.

Summary of part II

- In vacuum GR there are wormholes sources by negative tension rings carrying a negative energy. The ring encircles the wormhole throat. Solutions depend on a parameter σ.
- For $\sigma \neq 0$ the ring supports a power-law singularity of the Weyl tensor and a conical singularity of the Ricci tensor.
- For $\sigma \to 0$ the Weyl tensor vanishes, the geometry becomes locally flat, but there remains the conical singularity of the Ricci tensor corresponding to the negative energy $T = -c^4/(4G)$ along the ring. The ring "cuts a hole" in flat space.

III. Ring wormhole as the $M \to 0$ limit of Kerr spacetime

Minkowski space in spheroidal coordinates

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\varphi^2.$$

expressed in oblate spheroidal coordinates $r \in [0, \infty)$, $\vartheta \in [0, \pi)$

$$z = r \cos\vartheta, \qquad \rho = \sqrt{r^2 + a^2}\, \sin\vartheta$$

reads

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2 \cos^2\vartheta}{r^2 + a^2}\right)\left[dr^2 + (r^2 + a^2) d\vartheta^2\right] + (r^2 + a^2)\sin^2\vartheta d\varphi^2$$

Coordinate singularity at the ring $r = 0$, $\vartheta = \pi/2$. Geodesic

$$\frac{dr}{ds} = \pm\sqrt{\mathcal{E}^2 - \mu^2}$$

is discontinuous since one is bound to chose different signs.

Analytic continuation to $r \in (-\infty, \infty)$

If r is allowed to be negative – no need to change sign in geodesic equation; geodesics analytically continue. The metric is the same

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2\vartheta}{r^2 + a^2}\right)\left[dr^2 + (r^2 + a^2)d\vartheta^2\right] \\ +(r^2 + a^2)\sin^2\vartheta d\varphi^2\,,$$

and close to the ring $r = 0$, $\vartheta = \pi/2$ this reduces to

$$ds^2 = -dt^2 + dR^2 + R^2 d\alpha^2 + a^2 d\varphi^2 + \ldots$$

where $\alpha \in [0, 4\pi]$ hence the negative angle deficit and the distributional conical singularity of the curvature. The geometry can be covered by two flat charts (ρ_+, z_+) and (ρ_-, z_-)

$$ds^2 = -dt^2 + d\rho_\pm^2 + dz_\pm^2 + \rho_\pm^2 d\varphi^2$$

Wormhole topology

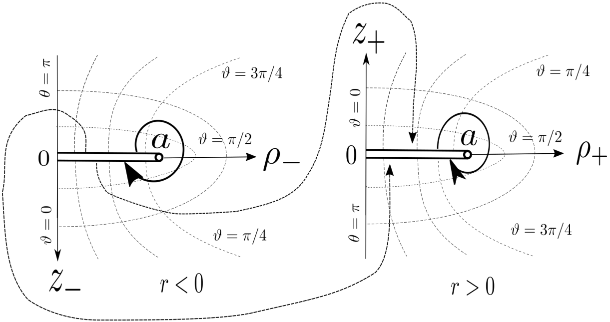


Figure: Analytic continuation from one flat chart to the other. A contour around the string core makes one revolution of 2π, then passes to the other chart, and only after the second revolution of 2π closes – the angle increment is 4π.

Moral

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2\vartheta}{r^2+a^2}\right)\left[dr^2 + (r^2+a^2)d\vartheta^2\right] \\ +(r^2+a^2)\sin^2\vartheta d\varphi^2$$

describes flat Minkowski space if $r \in [0, \infty)$ and locally flat wormhole if $r \in (-\infty, \infty)$. This is the $M \to 0$ limit of Kerr

$$\begin{aligned} ds^2 &= -dt^2 + \frac{2Mr}{\Sigma}\left(dt - a\sin^2\vartheta\, d\varphi\right)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) \\ &+ (r^2+a^2)\sin^2\vartheta d\varphi^2\,; \\ \Delta &= r^2 - 2Mr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\vartheta, \end{aligned}$$

but for Kerr $r \in (-\infty, \infty)$ since the geodesics pass to the $r < 0$ region.

Kerr geodesics

$$\frac{1}{\mu^2}\left(\frac{dr}{ds}\right)^2 + V(r) = E$$

As $M \to 0$ the geodesics freely move in $r \in (-\infty, \infty)$.

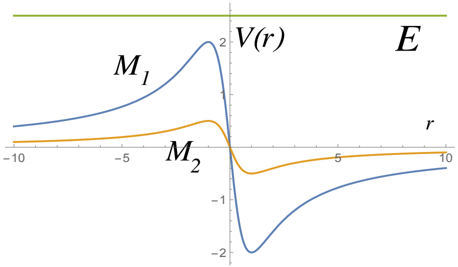


Figure: Potential $V(r) = -2Mr/(r^2 + a^2)$ in the geodesic equation

$\Rightarrow$ zero mass limit of Kerr is the wormhole

Kerr-Schild: $t, r, \vartheta, \varphi \to T, \rho, z, \varphi$

$$\begin{aligned}
\rho &= \sqrt{r^2 + a^2}\sin\vartheta, \quad z = r\cos\vartheta, \\
T &= t + \int \frac{2Mr}{\Delta}\,dr, \quad \phi = \varphi + \int \frac{2Mar}{\Sigma\Delta}\,dr,
\end{aligned}$$

which yields

$$\begin{aligned}
ds^2 = &- dT^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2 \\
&+ \frac{2Mr^3}{r^4 + a^2 z^2}\left(\frac{r\rho}{r^2 + a^2}\,d\rho - ar\sin^2\vartheta d\varphi + \frac{z}{r}\,dz + dT\right)^2
\end{aligned}$$

For $M \to 0$ the metric is flat. However, one needs two Kerr-Schild chars: (ρ_+, z_+) for $r > 0$ and (ρ_-, z_-) for $r < 0$. These two charts are glued together precisely as was shown before (Hawking-Ellis), hence for $M \to 0$ one obtains the two-sheeted wormhole topology and a conical singularity.

Fig.27 from Hawking-Ellis

ally incomplete at the ring singularity. However the only timelike and null geodesics which reach this singularity are those in the equatorial plane on the positive r side (Carter (1968a)).

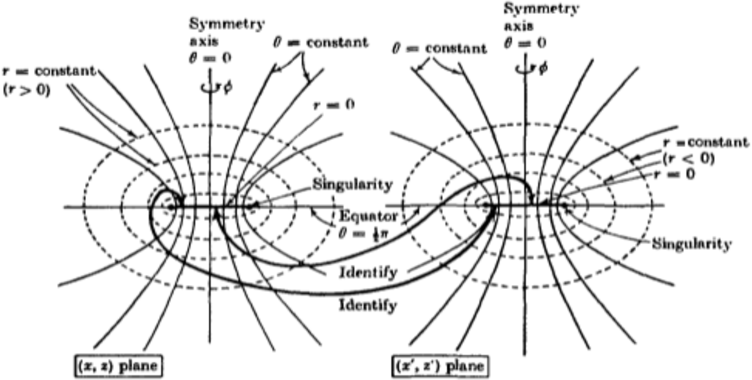


FIGURE 27. The maximal extension of the Kerr solution for $a^2 > m^2$ is obtained by identifying the top of the disc $x^2 + y^2 < a^2$, $z = 0$ in the (x, y, z) plane with the bottom of the corresponding disc in the (x', y', z') plane, and vice versa. The

Summary of part III

- Kerr spacetime has the two-sheeted topology also in the $M \to 0$ limit. The limiting spacetime is locally flat but it cannot be globally flat Minkowski space since it is topologically non-trivial.

- The Kerr ring supports a power-law singularity of the Weyl tensor that vanishes for $M \to 0$, but it also supports a distributional singularity of the Ricci tensor that remains even in the $M \to 0$ limit. Carter '68: *in the special case where M vanishes there must still be a curvature singularity at $\Sigma = 0$, although the metric is then flat everywhere else*.

- It follows that the $M \to 0$ limit of the Kerr spacetime is the wormhole sourced by the negative tension ring – the simplest way to produce wormholes.