Are Equations of Deep Water with a Free Surface Integrable?

V.E. Zakharov and A.I. Dyachenko

Novosibirsk State University, Novosibirsk, Russia; Lebedev Institute of Physics, Moscow;

Landau Institute for Theoretical Physics, Chernogolovka;

Department of Mathematics, University of Arizona, Tucson, USA;

Russian-French Workshop "Mathematical hydrodynamics", August 22-27, 2016, Novosibirsk, Russia

Basic equations

We study the potential flow of two-dimensional ideal incompressible fluid. The fluid occupies a half-infinite domain

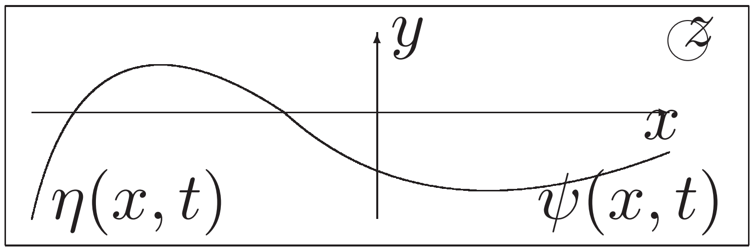
$$-\infty < y < \eta(x,t), \quad -\infty < x < \infty.$$

The flow is potential, so that $v = \nabla\Phi$, $\Phi|_{y=\eta(x,t)} = \psi(x,t)$. Boundary conditions on the surface are standard. It is known that the shape of surface $\eta(x,t)$ and the potential on the surface $\psi(x,t)$ form a pair of canonically conjugated variables obeying the Hamiltonian equations:

$$\frac{\partial\eta}{\partial t} = \frac{\delta\mathcal{H}}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\eta}.$$

Here $\mathcal{H}$ is Hamiltonian function, the total energy of the fluid.

Hamiltonian



$$H = \frac{1}{2}\int g\eta^2 + \psi\hat{k}\psi dx - \frac{1}{2}\int\{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + $$
$$+ \frac{1}{2}\int\{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx + \dots$$

$$\hat{k} = \sqrt{-\frac{\partial^2}{\partial x^2}}$$

Normal variables a_k

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \quad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}) \qquad \omega_k = \sqrt{gk}$$

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \dots$$

$$\begin{aligned} \mathcal{H}_2 &= \int \omega_k |a_k|^2 \\ \mathcal{H}_3 &= \mathcal{H}_3(a_k, a_k^*) - \text{third power} \\ \mathcal{H}_4 &= \mathcal{H}_4(a_k, a_k^*) - \text{fourth power} \end{aligned}$$

a_k satisfies the equation $\dfrac{\partial a_k}{\partial t} + i\dfrac{\delta H}{\delta a_k^*} = 0$

$$a_k \Rightarrow b_k$$

Canonical transformation excludes cubic terms. After transformation b_k satisfies the equation:

$$i\dot{b}_k = \omega_k b_k + \int \mathbf{T}_{kk_1}^{k_2k_3} \underline{b_{k_1}^* b_{k_2} b_{k_3}} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

Miracle #1

$$\mathbf{T}_{k_2k_3}^{kk_1} = \theta(kk_1k_2k_3)\mathbf{W}_{k_2k_3}^{kk_1}$$

In other words if $k_1, k_2, k_3 > 0,\ \ k < 0, \Rightarrow T_{kk_1}^{k_2k_3} \equiv 0!$

Let all $k_i > 0$. Then

$$\mathbf{T}^{kk_1}_{k_2k_3} = \frac{(kk_1k_2k_3)^{\frac{1}{4}}}{4\pi}\left[(kk_1)^{\frac{1}{2}} + (k_2k_3)^{\frac{1}{2}}\right]\mathrm{min}(k, k_1, k_2, k_3)\theta(kk_1k_2k_3)$$

One more canonical transformation makes possible to replace

$$\mathbf{T}^{k_2k_3}_{kk_1} \Rightarrow \tilde{T}^{k_2k_3}_{kk_1}$$

$$\tilde{T}^{k_2k_3}_{kk_1} = \frac{(kk_1k_2k_3)^{\frac{1}{2}}}{2\pi}\mathrm{min}(k, k_1, k_2, k_3)\theta(kk_1k_2k_3).$$

or

$$\begin{aligned}\tilde{T}^{k_2k_3}_{kk_1} &= \theta(kk_1k_2k_3)\frac{(kk_1k_2k_3)^{\frac{1}{2}}}{8\pi}(k + k_1 + k_2 + k_3 - \\ &- |k - k_2| - |k - k_3| - |k_1 - k_2| - |k_1 - k_3|)\end{aligned}$$

$$c_k = k^{\frac{1}{2}}\theta k b_k$$

$$\frac{\partial c}{\partial t} + i\hat{\omega}c - i\hat{P}^{+}\frac{\partial}{\partial x}\left(|c|^2\frac{\partial c}{\partial x}\right) = \hat{P}^{+}\frac{\partial}{\partial x}(\mathcal{U}c)$$

one can recognize two terms in the equation:

- nonlinear waves: $i\hat{\omega}c - i\hat{P}^{+}\frac{\partial}{\partial x}\left(|c|^2\frac{\partial c}{\partial x}\right) \Rightarrow$ EXTREAME WAVES

- advection term: $\hat{P}^{+}\frac{\partial}{\partial x}(\mathcal{U}c) \Rightarrow$ WAVE pre-BREAKING

$\mathcal{U} = \hat{K}|c|^2$ - advection velocity. $|c|^2$ - potential. $\hat{P}^{+}_k = \theta(k)$.

Breather is the localized solution of the following type:

$$c(x,t) = C(x - Vt)e^{i(k_0 x - \omega_0 t)} \quad or \quad c_k = e^{i(\Omega + Vk)t}\phi_k$$

where ϕ_k satisfies the equation:

$$(\Omega + Vk - \omega_k)\phi_k = \frac{1}{2}\int T_{kk_1}^{k_2k_3}\phi_{k_1}^*\phi_{k_2}\phi_{k_3}\delta_{k+k_1-k_2-k_3}dk_1dk_2dk_3$$

It can be found by Petviashvili method

$$\begin{aligned} \phi_k^{n+1} &= \frac{NL_k^n}{M_k}\left[\frac{< \phi^n \cdot NL(\phi^n) >}{\phi^n \cdot M\phi^n}\right]^\gamma, \quad M_k = \Omega + Vk - \omega_k, \\ NL(\phi^n) &= -P^+\frac{\partial}{\partial x}\left(|\phi^n|^2\frac{\partial\phi^n}{\partial x}\right) + iP^+\frac{\partial}{\partial x}\left(\hat{k}\left(|\phi^n|^2\right)\phi^n\right) \end{aligned}$$

Giant Breather

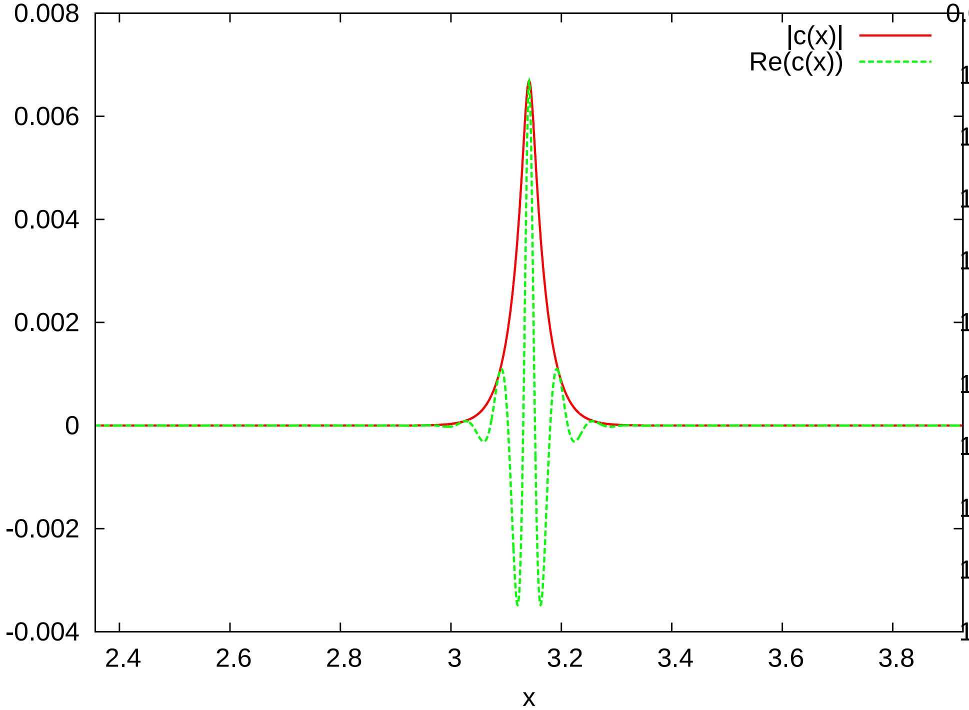


Figure 1: $|c(x)|$ (red curve)

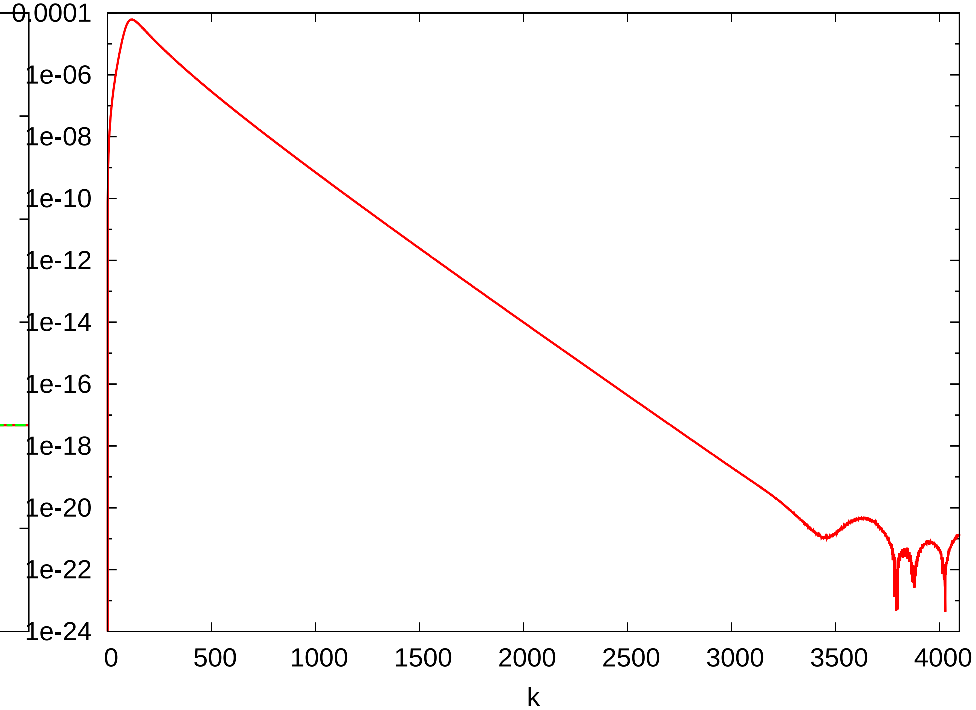


Figure 2: Spectrum $|c(k)|$)

Breathers collision

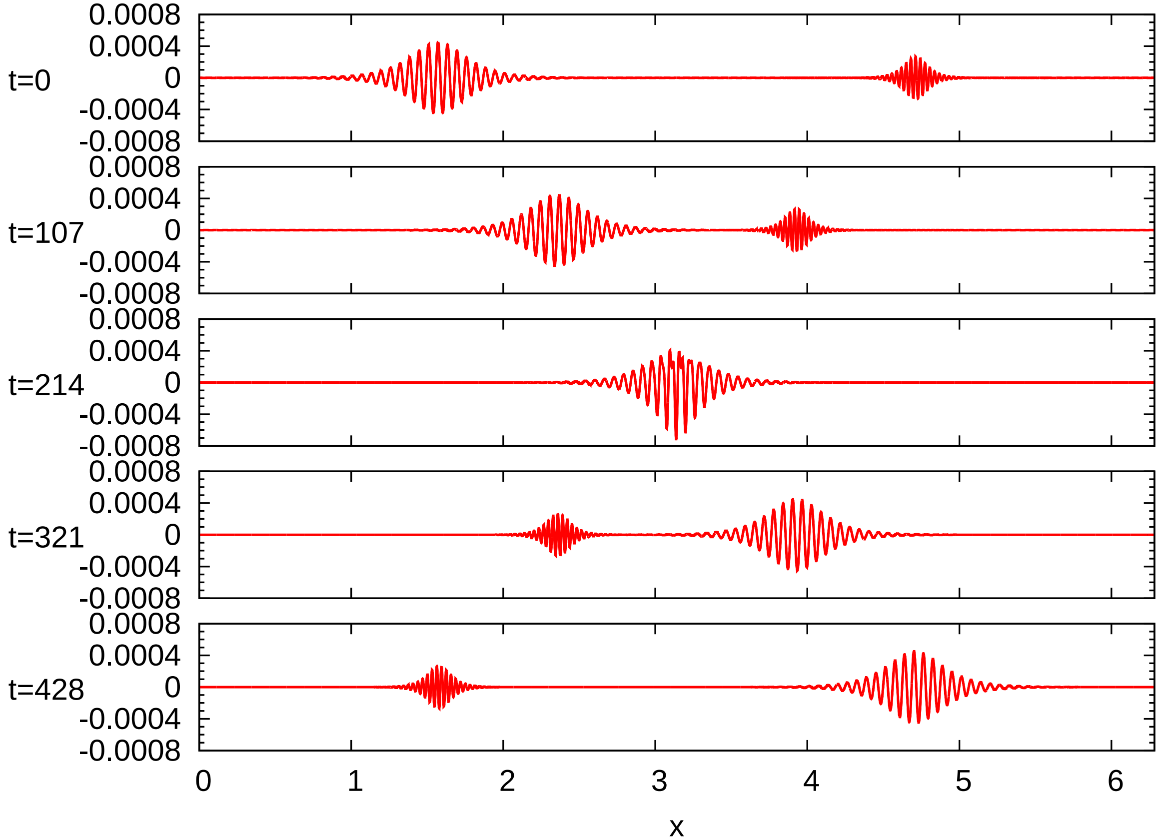


Figure 3: Collision of two breathers. Free surface for different times $t = 0, 107, 214, 321, 428$

Modulation Instability of Stokes Wave → Freak Wave

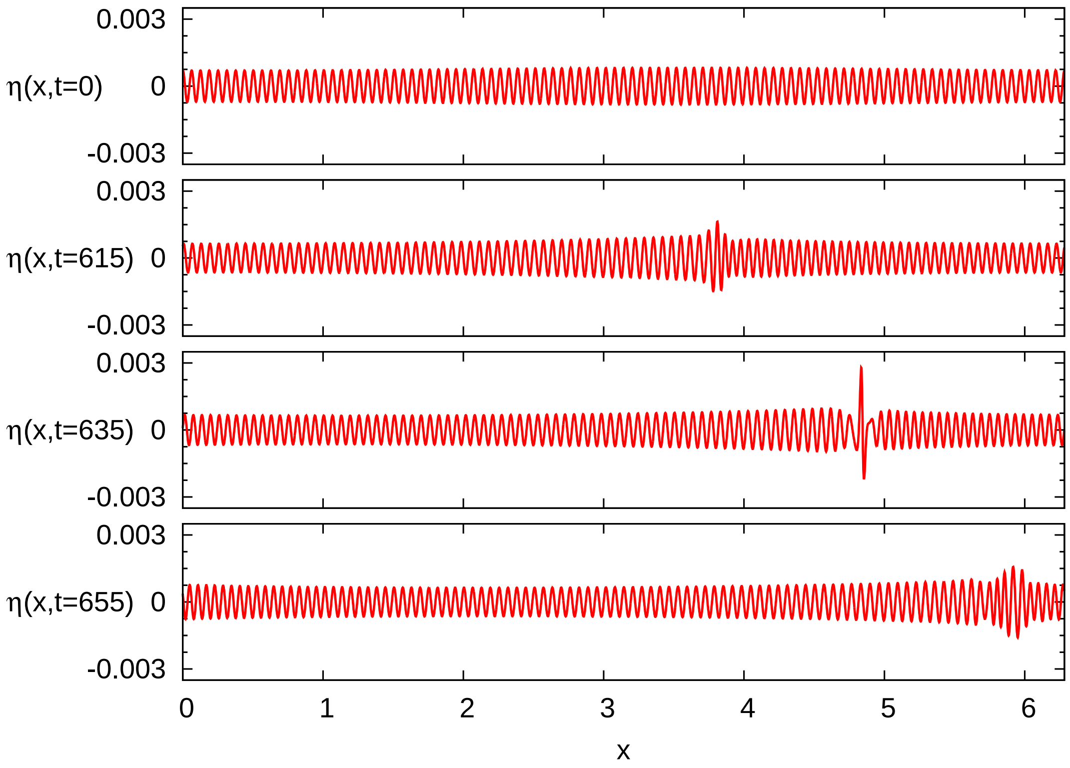
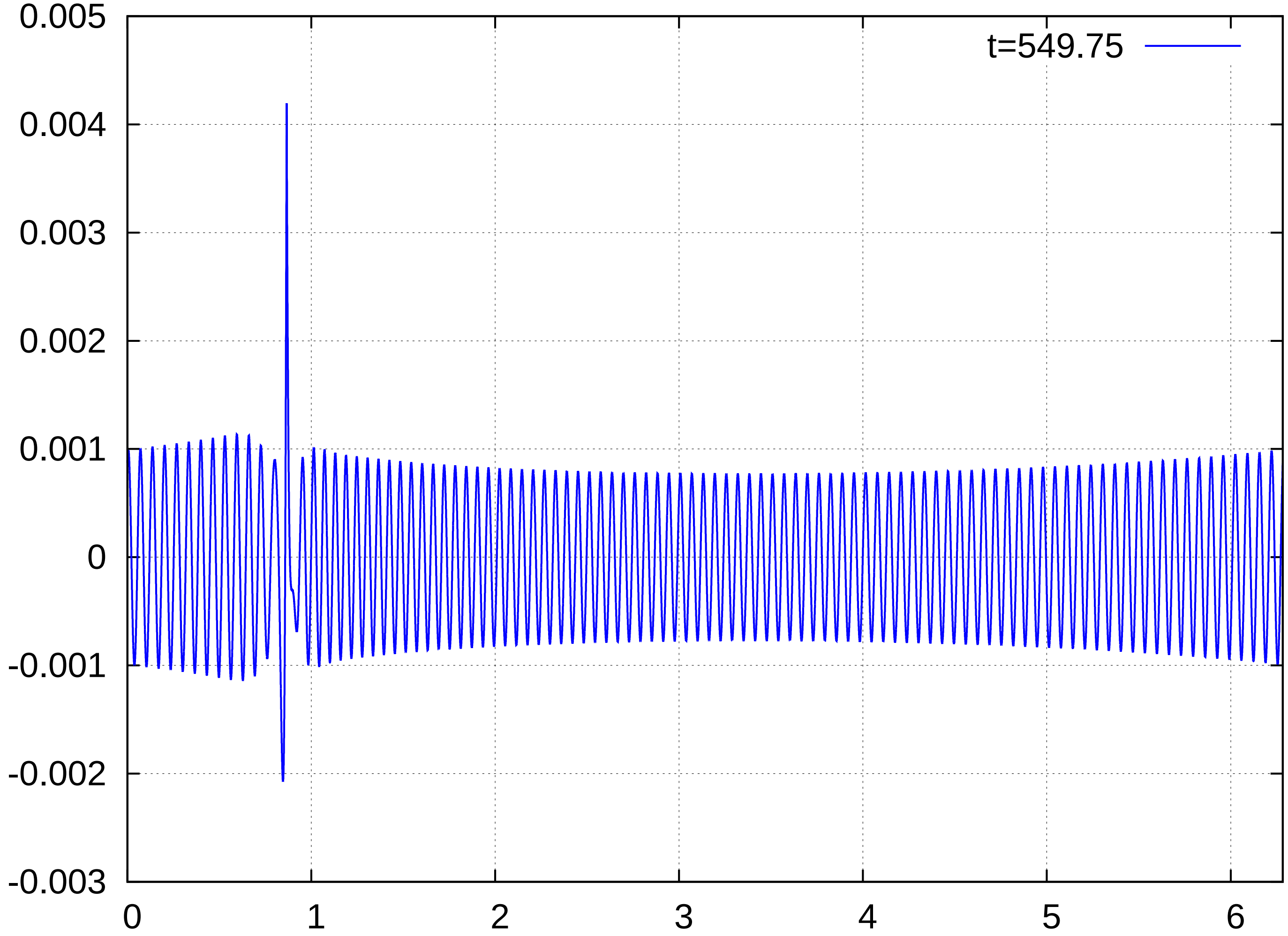


Figure 4: Formation of the freak wave. Free surface for different times $t = 0, 615, 635, 655$

Wave breaking



Euler equation in conformal variables

These equations minimize the action

$$S = \int L dt, \qquad L = \int_{-\infty}^{\infty} \psi \eta_t dx - \mathcal{H}.$$

Starting from this point let us forget for a while about hydrodynamics, and consider more general case. Namely, let's think of $\mathcal{H}$ as some arbitrary functional of ψ and η.

Let $z(w, t)$ be the conformal mapping of the domain, bounded by the curve $\eta(x, t)$ to the lower half-plane of w

$$w = u + iv, \qquad -\infty < u < \infty, \quad -\infty < v < 0$$

We introduce two functions analytic in the lower half-plane

$$z = x + iy = z(w)$$

$$\Phi = \Psi + i\hat{H}\Psi$$

These complex-valued functions are analytic in the lower half-plane $v \leq 0$.

Equations for "implicit" equations of motion can be rewritten as follows:

$$z_t\bar{z}_u - \bar{z}_t z_u = -\Phi_u + \bar{\Phi}_u$$

$$\Psi_t z_u - \Psi_u z_t + \frac{1}{2}\frac{\bar{\Phi}_u^2}{\bar{z}_u} = 0$$

$$\Psi = \frac{1}{2}(\Phi + \bar{\Phi})$$

Self-similar compressed fluid

$$\eta \equiv 0$$

$$\Phi(x, y, t) = \frac{1}{2}\frac{1}{t - t_0}(x^2 - y^2)$$

$$P = -\frac{y^2}{(t - t_0)^2} \qquad P = 0, y = 0$$

In conformal variables

$$z_0 = tu \qquad \Phi_0 = \frac{1}{2}tu^2$$

Then equations for the shape of self-similar solutions are satisfied. Let us study perturbation of this solution

$$z \to ut + z \qquad \Phi \to \frac{1}{2}u^2 t + \Phi$$

Equations for the self-similar solutions read

$$tz_t - uz_u + \Phi_u = P^-(\bar{z}_t z_u - z_t \bar{z}_u)$$

$$P^- \left\{ \frac{u}{2}(uz_u - \Phi_u) + t(\frac{1}{2}\Phi_t - uz_t) + \Psi_t z_u - \Psi_u z_t \right\} = 0$$

Miracle # 2

These solutions are satisfied if

$$z = \alpha(u) \qquad \Phi = \Phi_0(u) = \partial^{-1} u\alpha(u)$$

$\alpha(u)$ is an arbitrary! function analytic in the lower half-plane

$$\alpha(w) \to 0 \qquad Im w \to -\infty$$

Let

$$\alpha = \frac{A}{u + ia} \qquad A, a - real\ \ constants, u > 0$$

Shape of the surface is presented in the parametric form

$$x = u + \frac{Aut}{u^2 + a^2t^2} \qquad y = -\frac{aAt^2}{u^2 + a^2t^2}$$

$$\frac{\partial x}{\partial u} \to 1 \qquad at \qquad t \to \pm\infty$$

Bifurcation condition $\partial z/\partial u = 0$ leads to expression

$$u^2 = \frac{1}{2}A\,t\Big(1 \pm \sqrt{1 - \frac{8a^2}{A^2}}\Big) - a^2t^2$$

If

$$a^2 > \frac{1}{8}A^2$$

the solution is one-valued.

If

$$a^2 < \frac{1}{8}A^2$$

ie, the pole is close to the real axis, we obtain invertible:

1. Formation of bubbles (if $A > 0$)

2. Formation of droplets (if $A < 0$)

The face of surface is symmetric

Miracle # 3

Let us look for solution of the above equations in the form

$$z = \alpha(u) + \frac{1}{t}\, z_1(u) + \frac{1}{t^2}\, z_2(u) + \cdots$$

$$\Phi = \Phi_0(u) + \frac{1}{t}\, \Phi_1(u) + \frac{1}{t^2}\, \Phi_2(u) + \cdots$$

Now again $z_1(u)$ is arbitrary function analytic in the lower half-plane

$$\Phi_1(u) = u\, z_1(u)$$

$$u\, z_2(u) = -P^- \left(\bar{z}_1 \alpha_u - z_1\, \bar{\alpha}_u\right)$$

The system is integrable!

Dyachenko equations

There is another form of complex equations. Following Dyachenko, we introduce new variables:

$$R = \frac{1}{z'}, \qquad V = i\frac{\partial \Phi}{\partial z} = iR\Phi'.$$

For the simplest case of absence of gravity the Dyachenko equations read

$$R_t = i(UR' - RU')$$

$$V_t = i(UV' - RB')$$

In R and V variables:

$$U = \hat{P}^{-}(R\bar{V} + \bar{R}V), \quad B = \hat{P}^{-}(V\bar{V})$$

In the presence of gravity the first equation is not changed.

The second one takes the form:

$$V_t = i\Big(UV' - R\hat{P}^-(V\bar{V})' + g(R-1)\Big)$$

Poles and cuts

Functions R, V, U, B are analytic on $Im\, w < 0$. Moreover, $R \neq 0$, $Jm\, w < 0$.

However these functions may have singularities on upper half-plane. Function R can have zeros at $Jmw > 0$.

The following facts are important:

1. Zeroes of R (denote them λ_n) are persistent: $R(\lambda_n) = 0$. They cannot appear or disappear and move obeying the law

$$\dot{\lambda}_n = i\, U_n, \qquad U_n = U|_{w=\lambda_n}$$

2. Cuts are persistent if they are of root square type.

Motion constants

We see that approximation of narrow cut leads to an integrable system. Is the whole system integrable? The Dyachenko equations can be rewritten in the differential form

$$\frac{\partial}{\partial t}\frac{1}{R} = i\frac{\partial}{\partial w}\left(\frac{U}{R}\right), \qquad \frac{\partial}{\partial t}\frac{V}{R} = i\frac{\partial}{\partial w}\left(\frac{UV}{R} - B\right) + g\left(1 - \frac{1}{R}\right)$$

Let $I = \int_{-\infty}^{\infty} \frac{1}{R}\, du$, $J = \int_{-\infty}^{\infty} \frac{V}{R}\, du$. Then

$$\frac{dI}{dt} = 0, \qquad \frac{dJ}{dt} = -gI,$$

and $I = const$, $J = J_0 - gIt$. These equalities are conservation laws of mass and horizontal component of momentum. However, these relations could be generalized.

Let Γ be a closed contour and all functions be analytic in some neighborhood of this contour,

$$I = \oint_\Gamma \frac{1}{R}\, dw, \qquad J = \oint_\Gamma \frac{V}{R}\, dw,$$

and $I,\ J_0$ be motion constants.

If in a vicinity of λ_n, R and V can be presented as follow

$$R = a_n(w - \lambda_n) + \cdots \qquad V = b_n + b_1(w - \lambda_n) + \cdots$$

then

$$\begin{aligned} \frac{da_n}{dt} &= 0 \qquad a_n = const \\ \frac{db_n}{dt} &= -ga_n \qquad b_n = b_{0n} - ga_n t \end{aligned}$$

In other words, a_n, b_{0n} are motion constants. We conclude that each zero of R generates two complex (four real) motion constants.

All motion integrals are in involution. They form the Abelian Lie algebra. The question about the completeness of the set of integrals is open yet.

All functions R, V, U, B can be analytically continued to a certain Riemann surface, and each list of this surface generates additional motion constants.

This fact leads to the plausible conjecture that the whole set of motion constants is complete, hence the system is completely integrable.

The fact of integrability of the "compressed fluid" supports this conjecture. But this is just a conjecture yet. Anyway existence of extra motion constants is a **Miracle # 4**.

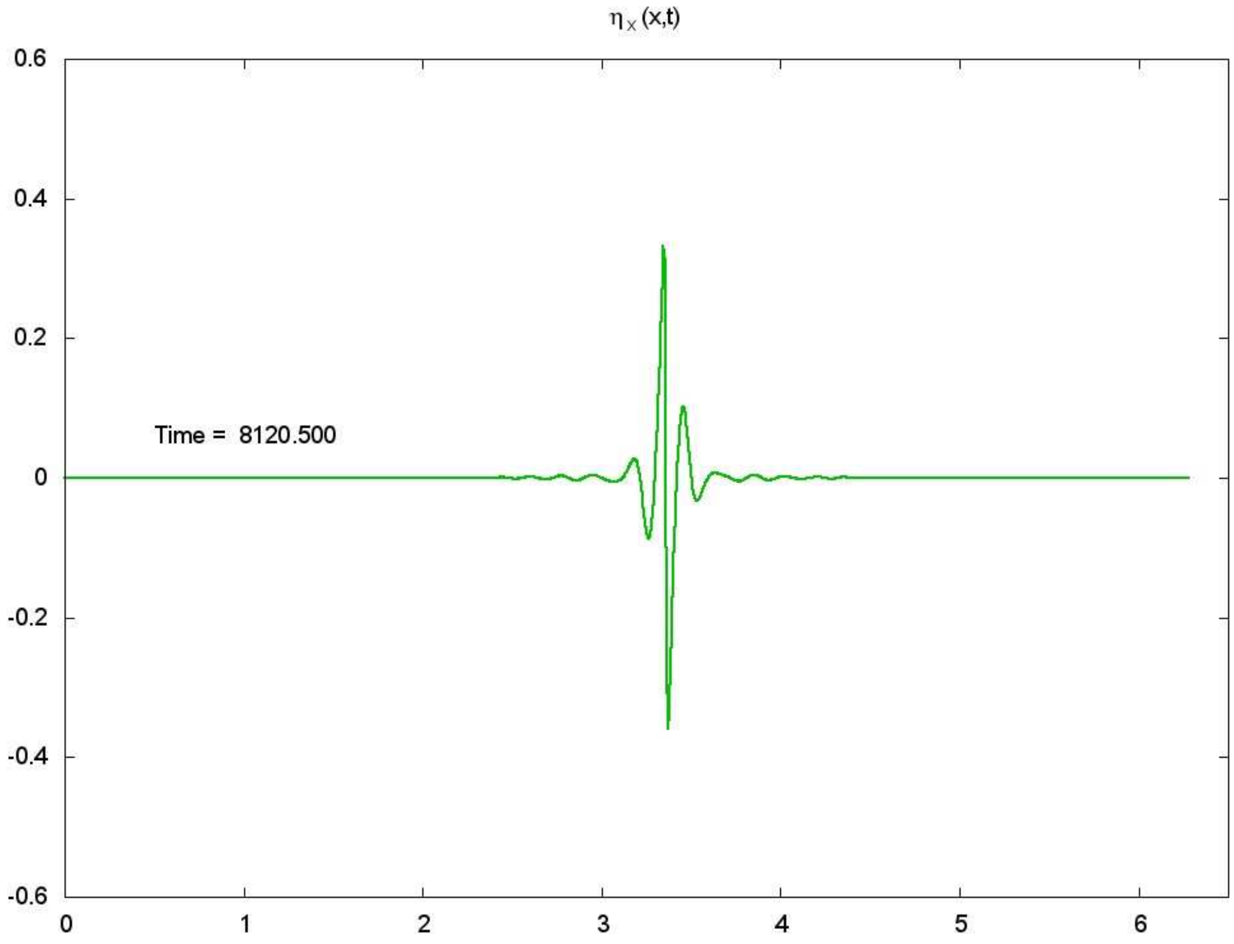
"Eternal" breather as a solution of exact equations

The compact equations have a solution in a form of breather propagating without radiation. Do exact Euler equation have a similar solution - "the eternal breather"? This is the open question. Theoretically speaking, any brether must loose energy due to radiation in the backward direction. If this radiation is absent, this is

Miracle # 5

Our numerical experiment supports existence of "the eternal breather".

Breather in the fully nonlinear (exact) equations (steepness).



Self-similar solutions

Equations

$$z_t\bar{z}_u - \bar{z}_t z_u = -\Phi_u + \bar{\Phi}_u$$
$$\Psi_t z_u - \Psi_u z_t + \frac{1}{2}\frac{\bar{\Phi}_u^2}{\bar{z}_u} = 0$$

admit the following substitution

$$z = t^\alpha z_0(u)$$
$$\Phi = t^{2\alpha-1}\Phi_0(u)$$

Then, the self-similar solutions are

$$\eta = t^\alpha F(\frac{x}{t^\alpha}), \qquad t \to t - t_0$$

In the presence of gravity only one solution is possible, $\alpha = 2$

$$\eta = g(t_0 - t)^2 F(\frac{x}{g(t_0 - t)^2})$$

This is formation of wedge with $\alpha = 120^0$ (another talk). If $g = 0$, all α are possible

$$\alpha(z_0 \bar{z}_{0u}) = \bar{\Phi}_{0u} - \Phi_{0u}$$

$$(2\alpha - 1)\Psi_0 z_{0u} - \alpha \Psi_{0u} z + \frac{1}{2}\frac{\bar{\Phi}_u^2}{\bar{z}_u}$$

$$\Psi_0 = \frac{1}{2}(\Phi_0 + \bar{\Phi}_0) = Re\Phi_0$$

If $\alpha = -3$ parabolic Dirichlet jet

If $\alpha = -1$ compressed fluid Other cases are not explored

Self-similar solutions must be found numerically